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THE ASYMPTOTIC POINCARE LEMMA AND ITS APPLICATIONS*

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Abstract. An asymptotic version of Poincaré's lemma is defined and solutions are obtained with the calculus of exterior differential forms. They are used to construct the asymptotic approximations of multidimensional oscillatory integrals whose forms are commonly encountered, for example, in electromagnetic problems. In particular, the boundary and stationary point evaluations of these integrals are considered. The former is applied to the Kirchhoff representation of a scalar field diffracted through an aperture and simply recovers the Maggi–Rubinowicz–Miyamoto–Wolf results. Asymptotic approximations in the presence of other (standard) critical points are also discussed. Techniques developed for the asymptotic Poincaré lemma are used to generate a general representation of the Leray form. All of the (differential form) expressions presented are generalizations of known (vector calculus) results.

1. Introduction. Multidimensional integrals are encountered in many areas of physics and engineering. A combination of Poincaré's lemma and Stokes' theorem provides a means of reducing a multidimensional integral to a lower dimensional form, hence, constitutes an appealing approach to its evaluation. However, the expressions that represent solutions of Poincaré's lemma are cumbersome and often difficult to evaluate explicitly. Furthermore, in many practical problems (for instance, in electromagnetics at high frequencies) a large parameter is present and an asymptotic approximation of these integrals is quite adequate. An asymptotic version of Poincaré's lemma whose solutions are readily computed would render the Poincaré–Stokes approach very tractable in these cases.

In §2 the asymptotic Poincaré lemma (APL) is formulated, and its solutions are derived with the calculus of exterior differential forms [1]–[3]. (All differential form notation concurs with that defined in [1].) These results are utilized in §§3, 4 to construct, respectively, the boundary and stationary point approximations of a multidimensional oscillatory integral. The resultant differential form representations encompass the standard vector expressions given, for instance, in [4] and [5]. The boundary point technique is applied in §3 to the Kirchhoff representation of the diffraction of a scalar field by an aperture in a perfectly conducting screen. The Maggi–Rubinowicz–Miyamoto–Wolf expressions [6]–[8] and their properties are recovered. Several other critical point contributions are also considered in §4. The Leray form [9] is constructed in an appendix with the APL method of solution. This form is utilized in the asymptotic approach given in [10]. The results of this paper are summarized in §5.

2. Asymptotic Poincaré lemma. Consider on the domain X, a set diffeomorphic to some open set in $\mathfrak{R}^n$, a p-form of the type

$$e^{\nu\Gamma}\beta, \tag{2.1}$$

where over X the phase function Γ is smooth and real-valued and the amplitude p-form β is smooth and complex-valued. The constant ν equals ik, where k is a large real

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parameter. For electromagnetic (quantum mechanical) problems k is 2π divided by the wavelength λ: $k=2\pi/\lambda$ ($k=2\pi/h$, where h is Planck's constant). A $(p-1)$-form, $e^{\nu\Gamma}\alpha$, of the same type as (2.1) is desired such that

$$d(e^{\nu\Gamma}\alpha)=e^{\nu\Gamma}\beta. \tag{2.2}$$

From Poincaré's lemma [1]–[3] it is known that a solution, $e^{\nu\Gamma}\alpha$, of (2.2) can exist only if the p-form $e^{\nu\Gamma}\beta$ is *closed*, i.e., only if

$$d(e^{\nu\Gamma}\beta)=e^{\nu\Gamma}(\nu\kappa+d)\beta=0, \tag{2.3}$$

where

$$\kappa=d\Gamma. \tag{2.4}$$

(Obviously,

$$d\kappa=0, \tag{2.5}$$

hence, κ is closed.) Condition (2.3) is satisfied if

$$(\kappa+D)\beta=0, \tag{2.6}$$

where

$$D=\nu^{-1}d. \tag{2.7}$$

On the other hand, because

$$D(e^{\nu\Gamma}\alpha)=e^{\nu\Gamma}(\kappa+D)\alpha, \tag{2.8}$$

(2.2) is equivalently represented as

$$\nu(\kappa+D)\alpha=\beta. \tag{2.9}$$

An asymptotic solution, α, of (2.9), when condition (2.6) is satisfied asymptotically, is constructed as follows. The result will be an *asymptotic solution of Poincaré's lemma.*

Consider a differential p-form of the type (2.1) when β has an asymptotic expansion

$$e^{\nu\Gamma}\beta=e^{\nu\Gamma}(\beta_0+\nu^{-1}\beta_1+\nu^{-2}\beta_2+\cdots), \tag{2.10}$$

where the β_j are p-forms. It is *asymptotically closed to range m* if the expression

$$(\kappa+D)\beta=(\kappa+D)(\beta_0+\nu^{-1}\beta_1+\nu^{-2}\beta_2+\cdots) \tag{2.11}$$

has its first $(m+1)$ terms (ordered in decreasing powers of ν) equal to zero; i.e., if the $(m+1)$ equations

$$\begin{aligned}&\kappa\beta_0=0,\\&\kappa\beta_1+d\beta_0=0,\\&\cdots\\&\kappa\beta_m+d\beta_{m-1}=0\end{aligned} \tag{2.12}$$

are satisfied. When these conditions hold, it is possible to find a $(p-1)$-form with an asymptotic expansion of a similar type

$$e^{\nu\Gamma}\alpha=\nu^{-1}e^{\nu\Gamma}(\alpha_0+\nu^{-1}\alpha_1+\nu^{-2}\alpha_2+\cdots), \tag{2.13}$$

such that the first $(m+1)$ terms of $d(e^{\nu\Gamma}\alpha)$ reproduce the first $(m+1)$ terms of $e^{\nu\Gamma}\beta$. This means an α can be found so that

$$\begin{aligned} &\kappa\alpha_0=\beta_0, \\ &\kappa\alpha_1+d\alpha_0=\beta_1, \\ &\quad\cdots \\ &\kappa\alpha_m+d\alpha_{m-1}=\beta_m. \end{aligned} \tag{2.14}$$

The resulting $(p-1)$-form α, limited to terms of degree not greater than $(m+1)$ in ν^{-1},

$$\alpha=\nu^{-1}\sum_{j=0}^{m}\nu^{-j}\alpha_j, \tag{2.15}$$

is an *m-th range asymptotic solution of Poincaré's lemma.*

The relations (2.12) and (2.14), which specify an mth range asymptotic solution of Poincaré's lemma, are represented by the flow diagram given in Fig. 1. Each location is the sum of the contributions indicated by the arrows leading to it. The operator $\hat{\kappa}$ represents the exterior product by κ from the left:

$$\hat{\kappa}\colon \alpha\mapsto\kappa\alpha. \tag{2.16}$$

The fact that the equations at the $(p+1)$-form level are satisfied results from the identity

$$d\circ\hat{\kappa}+\hat{\kappa}\circ d=0, \tag{2.17}$$

a consequence of κ being closed and the Leibnitz derivative rule [1, (H.16)].

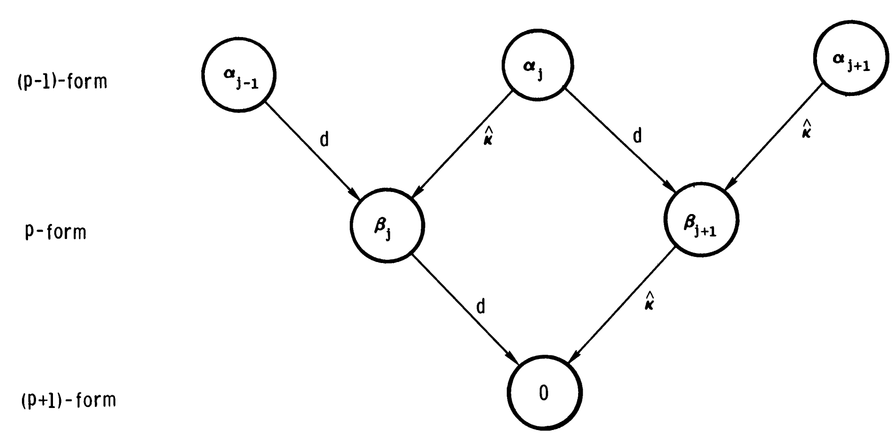

FIG. 1. *Relations that specify an asymptotic solution of Poincaré's lemma.*

Note that the expression (2.11) is automatically zero to any range if β is an n-form. Also, if the expression (2.10) consists of only the first term $e^{\nu\Gamma}\beta_0$; i.e., if $\beta=\beta_0$ and all other $\beta_j=0$, from (2.12) the terms of the expansion (2.13) of α are defined by the set of equations

$$\kappa\alpha_0=\beta,\quad \kappa\alpha_j=-d\alpha_{j-1},\quad \text{for } 1\le j\le m. \tag{2.18}$$

A solution of the system (2.14) when the conditions (2.12) are satisfied is based on the solution of an equation of the type

$$\kappa\alpha=\beta, \tag{2.19}$$

where the one-form κ and the p-form β are given and the $(p-1)$-form α is to be found. Its solution may be considered as a division of β by κ. Here α represents $\alpha_0, \alpha_1, \cdots, \alpha_m$ and correspondingly, β represents $\beta_0, \beta_1 - d\alpha_0, \cdots, \beta_m - d\alpha_{m-1}$. Multiplying (2.19) from the left by κ, one sees that a necessary condition for a solution to exist is that

$$\kappa\beta = 0. \tag{2.20}$$

Note that (2.19) and (2.20) are, respectively, the asymptotic (k large) approximations of (2.9) and (2.6). If $\kappa \neq 0$, let the one-form K be

$$K = (\kappa^*\kappa)^{-1}\kappa \equiv (\kappa\cdot\kappa)^{-1}\kappa = |\kappa|^{-2}\kappa. \tag{2.21}$$

The definitions of the star operator $*$ and the scalar product operation $\cdot$ are taken to be those given, respectively, in [1, Appendices E, F]. With (2.16) the operator "exterior product from the left by K" is simply

$$\hat{K} = (\kappa\cdot\kappa)^{-1}\hat{\kappa}. \tag{2.22a}$$

It is an operator of degree $+1$. Its adjoint, K^*, is the operator of degree -1 that equals

$$K^* = -*^{-1}\hat{K}*(-1)^p = (\kappa\cdot\kappa)^{-1}\left[-*^{-1}\hat{\kappa}*(-1)^p\right] \equiv (\kappa\cdot\kappa)^{-1}\kappa^*, \tag{2.22b}$$

when acting on a p-form. Applied to β it gives the $(p-1)$-form

$$K^*\beta = (\kappa\cdot\kappa)^{-1}\kappa^*\beta = \xi, \tag{2.23}$$

which is a solution of (2.19).

Proof. The operator κ^* satisfies the derivative property:

$$\kappa^*(\kappa\beta) = (\kappa^*\kappa)\beta - \kappa(\kappa^*\beta), \tag{2.24}$$

hence, the equivalent relation:

$$K^* \circ \hat{\kappa} + \hat{\kappa} \circ K^* = \mathrm{id}, \tag{2.25}$$

where id represents the identity operator. Consequently, (2.20) and (2.24) yield

$$\kappa(\kappa^*\beta) = (\kappa^*\kappa)\beta = (\kappa\cdot\kappa)\beta, \tag{2.26}$$

hence,

$$\kappa\xi = (\kappa\cdot\kappa)^{-1}\kappa(\kappa^*\beta) = (\kappa\cdot\kappa)^{-1}(\kappa\cdot\kappa)\beta = \beta. \tag{2.27}$$ □

The operator K^* is a (right) inverse of $\hat{\kappa}$, the operator product by κ; the solution ξ is an element of the kernel of κ^*: $\kappa^*\xi = 0$; i.e., $\kappa^*\kappa^* \equiv 0$. An interesting application of this inversion algorithm, the construction of the Leray form [9, §3.1], is given in Appendix A. The condition $\kappa \neq 0$ is satisfied except at those points in X at which the phase function Γ is stationary. Note, however, that this is only a sufficient condition. The p-form β may approach zero in regions where the operator K^* is singular (i.e., where $d\Gamma = \kappa = 0$) in such a manner that ξ given by (2.23) remains finite. This behavior is encountered in the stationary phase evaluation of an integral and will be discussed further in §4.

Consequently, solutions of (2.14) are

$$\begin{aligned} \alpha_0 &= K^*\beta_0, \\ \alpha_1 &= K^*(\beta_1 - d\alpha_0), \\ &\cdots \\ \alpha_m &= K^*(\beta_m - d\alpha_{m-1}). \end{aligned} \tag{2.28}$$

These relations are expressed more compactly as

$$\alpha_j = \sum_{p=0}^{j} \{-K^*d\}^{j-p}(K^*\beta_p). \tag{2.28'}$$

To justify that $\alpha_1, \cdots, \alpha_m$ are solutions, one must verify that conditions such as (2.20) apply to each of their equations; i.e., for α_j, that $\kappa(\beta_j - d\alpha_{j-1})$ is null.

Proof. From (2.17) and (2.12) one has, respectively, $\kappa d\alpha_{j-1} = -d(\kappa d_{j-1})$ and $\kappa\beta_j = -d\beta_{j-1}$. Thus, with (2.14)

$$\kappa(\beta_j - d\alpha_{j-1}) = d(\kappa\alpha_{j-1} - \beta_{j-1}) = -d(d\alpha_{j-2}) \equiv 0. \tag{2.29}$$

When $\beta = \beta_0$ (all $\beta_j = 0$ for $j > 0$) and $\kappa\beta = 0$, the solutions (2.28) and (2.28′) reduce to the expressions:

$$\begin{aligned} \alpha_0 &= K^*\beta, \\ \alpha_j &= -K^*d\alpha_{j-1} \quad \text{for } 1 \le j \le m, \end{aligned} \tag{2.30}$$

and

$$\alpha_j = \{-K^*d\}^j(K^*\beta) \quad \text{for } 0 \le j \le m. \tag{2.30'}$$

These results can be summarized with a statement of the

ASYMPTOTIC POINCARÉ LEMMA (APL): *If a given p-form $e^{\nu\Gamma}\beta$ is asymptotically closed to range m over a domain X, a set diffeomorphic to some open set in $\mathfrak{R}^n$, where $d\Gamma \neq 0$, it is asymptotically exact to range m over that domain; i.e., there exists a $(p-1)$-form $e^{\nu\Gamma}\alpha$ defined over X such that $d(e^{\nu\Gamma}\alpha) = e^{\nu\Gamma}\beta + O(\nu^{-(m+1)})$.*

The first $(m+1)$ terms of α are readily constructed from those of β with (2.28′). The construction is not valid in general at points where $d\Gamma = 0$. This restriction may be lifted for some particular β at some points where $d\gamma = 0$ as shown in §4.

The solution (2.23) of (2.19) is not unique. The general solution of (2.19) is actually

$$\alpha = \xi + \kappa\gamma, \tag{2.31}$$

where γ is an arbitrary $(p-2)$-form, since $\hat{\kappa} \circ \hat{\kappa} \equiv 0$. The expression (2.31) represents a gauge transformation of the solution (2.23). The freedom to include a gauge term, $\kappa\gamma$, in that solution occurs because (2.19) determines only the components of α "transverse" to κ. In particular, the operator $\hat{\kappa} \circ K^*$ is a projection operator that selects from the form β its component, $\kappa(K^*\beta)$, along κ; i.e., its "longitudinal" component. Hence, because the identity (2.25) means

$$\kappa(K^*\beta) + K^*(\kappa\beta) = \beta, \tag{2.32}$$

the projection operator $\{1 - \hat{\kappa} \circ K^*\}$ selects components transverse to κ. Therefore, from (2.19) one immediately obtains

$$K^*\beta = \{1 - \kappa \circ K^*\}\alpha = \xi; \tag{2.33}$$

i.e., the $(p-1)$-form ξ is the component of α transverse to κ. This is also reflected in the fact that ξ is an element of the kernel of K^*. Similarly, the solution, $e^{\nu\Gamma}\alpha$, of (2.2) is also not unique. If one replaces α by α':

$$\alpha \to \alpha' = \alpha + (\kappa + D)\gamma \tag{2.34}$$

where γ is any $(p-2)$-form, the equation

$$D(e^{\nu\Gamma}\alpha) = e^{\nu\Gamma}(\kappa + D)\alpha$$

is still satisfied since

$$(\hat{\kappa}+D)\circ(\hat{\kappa}+D) = (\hat{\kappa}+D)^2 \equiv 0. \tag{2.35}$$

The one-term asymptotic result (2.31) is clearly recovered in the limit $\kappa \to \infty$. The transformation $\alpha \to \alpha'$ of the general mth range APL solution given by

$$\begin{aligned} &\alpha_0 \to \alpha_0' = \alpha_0 + \kappa\gamma_0, \\ &\alpha_j \to \alpha_j' = \alpha_j + (\kappa\gamma_j + d\gamma_{j-1}) \quad \text{for } 1 \le j \le m \end{aligned} \tag{2.36}$$

may be considered as an *asymptotic gauge transformation* of that solution.

Proof. The equations the gauge transformed solutions satisfy:

$$\kappa\alpha_j' = \beta_j - d\alpha_{j-1}', \tag{2.37}$$

reduce to (2.14) through the following sequence of relations:

$$\begin{aligned} &\kappa\alpha_j' = \kappa(\alpha_j + \kappa\gamma_j + d\gamma_{j-1}) = \beta_j - d(\alpha_{j-1} + \kappa\gamma_{j-1} + d\gamma_{j-2}) = \beta_j - d\alpha_{j-1}', \\ &\kappa\alpha_j + \kappa d\gamma_{j-1} = \beta_j - d\alpha_{j-1} - d(\kappa\gamma_{j-1}), \\ &\kappa\alpha_j = \beta_j - d\alpha_{j-1}. \end{aligned}$$

□

The gauge transformation (2.36) can be obtained from (2.34) with the $(p-2)$-form

$$\gamma = \nu^{-1}\sum_{j=0}^{m} \nu^{-j}\gamma_j \tag{2.38}$$

and with a truncation of the resultant expression for α' to degree $(m+1)$ in ν^{-1}. Thus, the asymptotic gauge transformation (2.36) is exact if γ_m is closed $(d\gamma_m = 0)$.

Now consider the case where the p-form (2.1) is given exactly by the $(m+1)$-term expression:

$$e^{\nu\Gamma}\beta = e^{\nu\Gamma}(\beta_0 + \nu^{-1}\beta_1 + \cdots + \nu^{-m}\beta_m). \tag{2.39}$$

It is of interest to determine when the APL solution is actually an exact solution. Recall that an exact solution can exist only if the p-form (2.39) is closed. The equivalent condition (2.6) is satisfied if, in addition to β being asymptotically closed, β_m is closed:

$$d\beta_m = 0. \tag{2.40}$$

Thus, if (2.12) and (2.40) are satisfied, there exists (locally) a $(p-1)$-form whose differential is $e^{\nu\Gamma}\beta$. This form is not necessarily given by the APL algorithm which searches for a particular expression $e^{\nu\Gamma}\alpha$, α given by (2.15) and (2.28′). However, if it is, the relation (2.9) requires

$$d\alpha_m = 0 \tag{2.41}$$

in addition to the satisfaction of (2.14). Since the differential of the relation $\beta_m = \kappa\alpha_m + d\alpha_{m-1}$ gives

$$d\beta_m = -\kappa d\alpha_m, \tag{2.42}$$

(2.40) is satisfied if (2.41) is. Conversely, (2.40) and (2.42) only imply that

$$\kappa d\alpha_m = 0. \tag{2.43}$$

Hence, the fact that (2.39) is exactly closed does not imply that the APL solution is exact. With (2.28′) the condition (2.41) can also be represented as

$$d\alpha_m = d\left[\sum_{p=0}^{m} (-K^*d)^{m-p}(K^*\beta_p)\right] = \sum_{p=0}^{m} (-1)^{m-p}(d\circ K^*)^{m-p+1}\beta_p = 0. \tag{2.44}$$

Let the operator

$$\hat{t} = -d\circ K^*. \tag{2.45}$$

Therefore, if

$$\sum_{p=0}^{m} \hat{t}^{\,m-p+1}\beta_p = 0 \tag{2.46}$$

the APL solution is exact.

The conditions for exactness when $m=0$, 1 will be employed in the next section. From (2.46) they are, respectively,

$$\hat{t}\beta_0 = 0, \tag{2.47}$$

$$\hat{t}(\hat{t}\beta_0 + \beta_1) = 0. \tag{2.48}$$

The $m=1$ condition (2.48) is satisfied if $\beta_1 = -\hat{t}\beta_0 = d\alpha_0$. In fact, since

$$\sum_{p=0}^{m} \hat{t}^{\,m-p+1}\beta_p = \hat{t}(\beta_m - d\alpha_{m-1}), \tag{2.49}$$

the condition for exactness (2.46) is satisfied in the general case if $\beta_m = d\alpha_{m-1}$. In the $m=0$ case (2.12) and (2.40) are satisfied if the p-form $\beta = \beta_0$ is longitudinal and closed:

$$\kappa\beta_0 = 0, \qquad d\beta_0 = 0, \tag{2.50}$$

conditions which are automatically satisfied if β_0 is an n-form. Subsequently, if V is the vector-field associated to the one-form K [1, App. E], (2.47) can also be written as

$$\mathfrak{L}_V\beta_0 = 0 \tag{2.51}$$

where the Lie derivative [1, App. L]

$$\mathfrak{L}_V = d\circ K^* + K^*\circ d. \tag{2.52}$$

Thus, if β_0 is invariant along the flow defined by V [1, App. L] in the $m=0$ case, the APL algorithm is exact.

Figure 2 summarizes (2.12), (2.14) and (2.36), i.e., the expressions defining the general APL solution. It extends Fig. 1 by including the terms that can be added to the α's as expressed by (2.36). Parallel arrows designate the same operation $\hat{k}$ or d; each

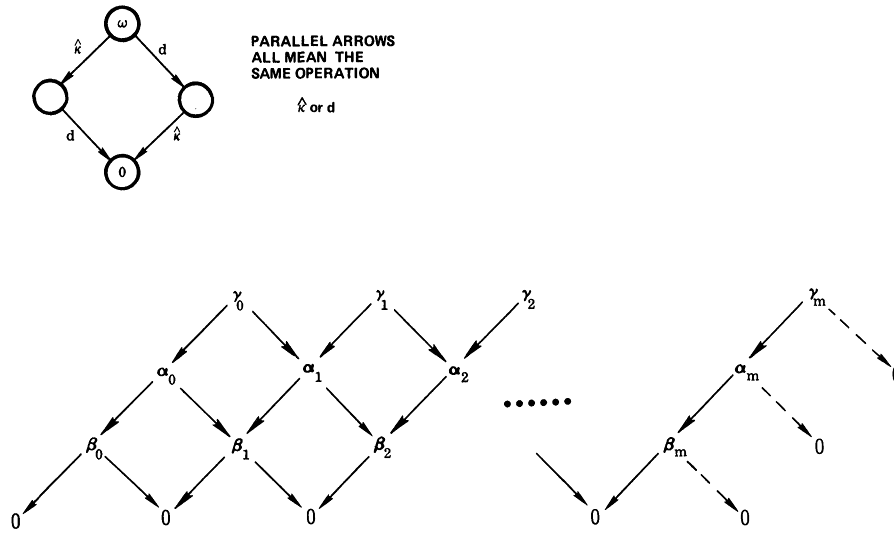

FIG. 2. *General relations characterizing an asymptotic solution of Poincaré's lemma.*

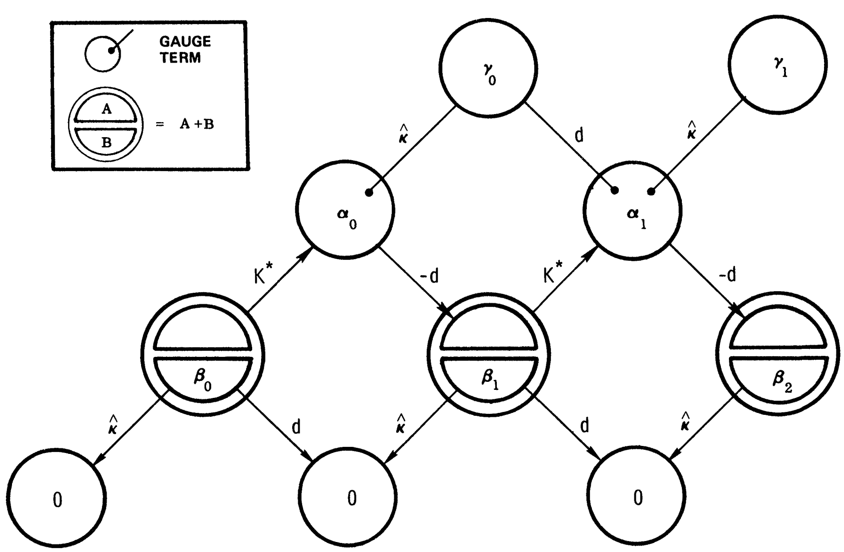

FIG. 3. *The general asymptotic solution of Poincaré's lemma.*

diamond pattern represents the identity (2.17). The dotted arrows on the right express the asymptotic satisfaction of those relations. Since they represent the expression $d\gamma_m=0$ and (2.41) and (2.40), they also are the conditions which describe when the APL algorithm is exact. Similarly, the diagram given in Fig. 3 summarizes (2.12), (2.28) and (2.36), i.e., the general APL solution. The lines ending in solid dots represent the gauge terms of (2.36). Half-moon elements are added before the operator leading from the circle surrounding them is applied. When $\beta=\beta_0$, this diagram simplifies to the one shown in Fig. 4 which summarizes (2.30) and (2.36) and the condition $\kappa\beta=0$. Note that with the representation of the operator $-K^* \circ d$ by the dotted lines, the diagram

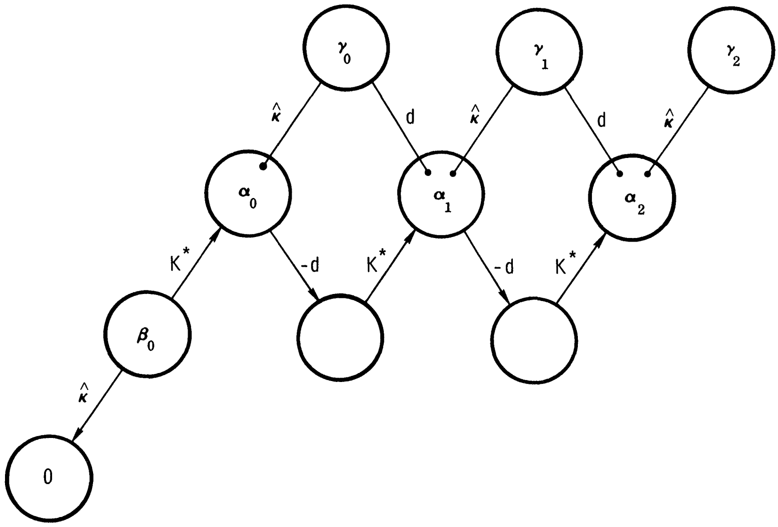

FIG. 4. *The asymptotic solution of Poincaré's lemma when $\beta=\beta_0$.*

suggests a geometrical progression. In particular, when $m=\infty$, the $(p-1)$-form α can be written as

$$\alpha=\nu^{-1}\sum_{j=0}^{\infty}\nu^{-j}(-K^*\circ d)^j\alpha_0=\nu^{-1}[1-\hat{s}]^{-1}\alpha_0, \tag{2.53}$$

where the operator

$$\hat{s}=\nu^{-1}(-K^*\circ d)\equiv -K^*\circ D. \tag{2.54}$$

3. Boundary point evaluation of an integral.

3A. General formalism. The contributions to the (asymptotic) evaluation of the integral

$$\int_{\mathfrak{D}} e^{\nu\Gamma}\beta, \tag{3.1}$$

also denoted (see [1, p. 677])

$$e^{\nu\Gamma}\beta|\mathfrak{D}, \tag{3.1$'$}$$

from the points on the boundary, Σ, of $\mathfrak{D}$ are desired. The domain $\mathfrak{D}$ is an oriented p-domain of finite extent in X; its boundary $\Sigma=\partial\mathfrak{D}$. If (2.2) holds in $\mathfrak{D}$, (3.1) can be evaluated immediately with Stokes' formula [1, App. J] so that

$$e^{\nu\Gamma}\beta|\mathfrak{D}=d(e^{\nu\Gamma}\alpha)|\mathfrak{D}=e^{\nu\Gamma}\alpha|\partial\mathfrak{D}=e^{\nu\Gamma}\alpha|\Sigma. \tag{3.2}$$

The desired asymptotic results follow in a similar fashion from the APL.

It is assumed that Γ has no stationary points in $\mathfrak{D}$; i.e., that $\kappa\neq 0$ in $\mathfrak{D}$ so that K, hence, α is not singular. The given p-form $e^{\nu\Gamma}\beta$ can be replaced with an exact differential of a $(p-1)$-form $e^{\nu\Gamma}\alpha$ that is a range m APL solution plus a remainder:

$$e^{\nu\Gamma}\beta=d\left[\nu^{-1}\left(e^{\nu\Gamma}\sum_{j=0}^{m}\nu^{-j}\alpha_j\right)\right]-\nu^{-(m+1)}e^{\nu\Gamma}(d\alpha_m). \tag{3.3}$$

Consequently, with Stokes' formula the integral relation corresponding to (3.3) is

$$e^{\nu\Gamma}\beta|\mathfrak{D}=\nu^{-1}\sum_{j=0}^{m}\nu^{-j}\left(e^{\nu\Gamma}\alpha_j|\Sigma\right)-\nu^{-(m+1)}\left(e^{\nu\Gamma}d\alpha_m|\mathfrak{D}\right). \tag{3.4}$$

The original integral (3.1) over $\mathfrak{D}$ has been replaced by an $(m+1)$-term asymptotic series defined over the boundary, Σ, of $\mathfrak{D}$ and a remainder term defined over $\mathfrak{D}$ that is of degree $(m+1)$ in ν^{-1}. Because this latter term is the same type as the original, it can be evaluated in a similar manner. Hence, it is asymptotically small compared to every other term in the sum. The desired (asymptotic) boundary point evaluation of the integral (3.1) is, therefore,

$$e^{\nu\Gamma}\beta|\mathfrak{D}\sim\nu^{-1}\sum_{j=0}^{m}\nu^{-j}\left(e^{\nu\Gamma}\alpha_j|\Sigma\right)\equiv e^{\nu\Gamma}\alpha|\Sigma. \tag{3.5}$$

If $m=0$, this reduces to

$$e^{\nu\Gamma}\beta|\mathfrak{D}\sim\nu^{-1}\left(e^{\nu\Gamma}\alpha_0|\Sigma\right). \tag{3.6}$$

Equation (3.5) is a generalization of the expressions given in [4, Chap. 8]. It is applicable to the vector as well as the scalar case. Furthermore, if the domain $\mathfrak{D}$ is one-dimensional, (3.5) reduces to the standard endpoint evaluation of an integral given, for example, in [11] or [12]. The following examples demonstrate the utility of the APL-boundary point analysis.

3B. Kirchhoff approximation. Let the space $\mathfrak{R}^3$ be divided into two regions V_1 and V_2 separated by an oriented surface $M=\partial V_1$, whose normal points toward V_2. The surface M is composed of a screen S and of an aperture $\mathfrak{D}$ (whose edge is $\Sigma=\partial\mathfrak{D}$) such that $M=\mathfrak{D}\cup S$ and is assumed to lie on side V_2 of the screen [1, Fig. 2]. Consider in the absence of the screen two scalar field solutions u_j $(j=1,2)$ of the Helmholtz equation whose sources ρ_j are in V_j

$$\{\Delta+\kappa^2\}u_j=\rho_j. \tag{3.7}$$

As discussed in [1, §IV], if U_1 is the field due to ρ_1 in the presence of the screen and if ρ_2 is a point source at $\mathbf{s}_2$, the field U_1 at $\mathbf{s}_2$ can be represented in terms of the cross-flux of U_1 and u_2 through M as

$$U_1(\mathbf{s}_2)=(U_1*du_2-u_2*dU_1)|M. \tag{3.8}$$

The Kirchhoff approximation assumes that the field U_1 and its derivative along the normal of M (represented by the term $*dU_1$) are zero over S and are equal, respectively, to u_1 and its normal derivative over $\mathfrak{D}$. The resultant representation of the field (3.8) is

$$U_1(\mathbf{s}_2)\cong\beta_{12}|\mathfrak{D}\equiv(u_1*du_2-u_2*du_1)|\mathfrak{D}. \tag{3.9}$$

The boundary point analysis will be applied to the integral $\beta_{12}|\mathfrak{D}$ in several cases. The results represent a reduction of the Kirchhoff approximation of the field (3.9) to a line integral over the edge of the aperture.

1. *Plane wave-plane wave case*. Consider the plane waves $u_j(\mathbf{r})=\exp[\nu(\kappa_j|\mathbf{r})]$, where $(\kappa_j|\mathbf{r})$ is the duality product (see [1, App. D]) of κ_j, a constant unit propagation one-form (i.e., $d\kappa_j=0$ and $\kappa_j\cdot\kappa_j\equiv\kappa_j|\kappa_j\equiv\kappa_j^*\kappa_j=1$), and the position vector $\mathbf{r}$. In Cartesian coordinates, for example, $\kappa_j=\xi_j dx+\eta_j dy+\zeta_j dz$ and $\mathbf{r}=x\partial_x+y\partial_y+z\partial_z$ so that

$\kappa_j|\mathbf{r}=\xi_j x+\eta_j y+\zeta_j z$. Since

$$du_j=\nu\kappa_j u_j, \tag{3.10}$$

the cross-flux two-form

$$\beta_{12}=\nu u_1 u_2 * (\kappa_2-\kappa_1)\equiv\nu(e^{\nu\Gamma}\beta), \tag{3.11}$$

where

$$\Gamma=(\kappa_1+\kappa_2)|\mathbf{r} \tag{3.12}$$

and

$$\beta=*(\kappa_2-\kappa_1). \tag{3.13}$$

One verifies that $\kappa\beta\equiv 0$ since $\kappa=d\Gamma=\kappa_1+\kappa_2$. The APL solution is

$$\alpha_0=K^*\beta=-\frac{*\kappa_1\kappa_2}{(1+\kappa_1\cdot\kappa_2)}, \tag{3.14}$$

all other $\alpha_j=0$ ($j=1,2,\cdots$) because α_0 is a constant. Thus, if

$$\alpha_{12}=e^{\nu\Gamma}\alpha_0=-u_1u_2\left(\frac{*\kappa_1\kappa_2}{1+\kappa_1\cdot\kappa_2}\right), \tag{3.15}$$

the cross-flux integral

$$\beta_{12}|\mathfrak{D}=\alpha_{12}|\Sigma. \tag{3.16}$$

Moreover, $d\beta=0$ (β is a constant); therefore, from the preceding section one realizes that (3.16) represents an exact result (hence, the equal sign). Note that (3.16) must be modified when $\kappa=0$; i.e., when $\kappa_1=-\kappa_2$, hence, when $\Gamma, \kappa_1\kappa_2$ and $(1+\kappa_1\cdot\kappa_2)$ are all zero.

2. *Spherical wave-spherical wave case*. Consider the two fields $u_i(\mathbf{r})=G(\mathbf{r}-\mathbf{s}_i)$ ($i=1,2$) due to point sources at $\mathbf{s}_1$ and $\mathbf{s}_2$. These fields represent spherical waves originating at those points. The Green's function

$$G(\mathbf{r})=\frac{\exp(\nu r)}{4\pi r}, \tag{3.17}$$

where $r=|\mathbf{r}|$. If $r_i=|\mathbf{r}-\mathbf{s}_i|$ and $\kappa_i=dr_i$, the cross-flux two-form

$$\beta_{12}=u_1u_2*\left[\left(\nu-\frac{1}{r_2}\right)\kappa_2-\left(\nu-\frac{1}{r_1}\right)\kappa_1\right]=\nu(e^{\nu\Gamma}\beta), \tag{3.18}$$

where the phase

$$\Gamma=r_1+r_2, \tag{3.19}$$

so that

$$\kappa=d\Gamma=\kappa_1+\kappa_2 \tag{3.20}$$

and the two-forms

$$\beta_0=g_1g_2*(\kappa_2-\kappa_1) \tag{3.21}$$

and

$$\beta_1 = -g_1 g_2 * \left(\frac{\kappa_2}{r_2} - \frac{\kappa_1}{r_1}\right), \tag{3.22}$$

so that $\beta = \beta_0 + \nu^{-1}\beta_1$. The functions

$$g_i(\mathbf{r}) = \frac{1}{4\pi r_i}. \tag{3.23}$$

The corresponding APL solution $\alpha = \nu^{-1}(\alpha_0 + \nu^{-1}\alpha_1)$ is obtained from the system

$$\kappa\alpha_0 = \beta_0, \tag{3.24a}$$

$$\kappa\alpha_1 = \beta_1 - d\alpha_0. \tag{3.24b}$$

Since $\kappa\beta_0 = 0$, (3.24a) is satisfied by

$$\alpha_0 = \frac{(\kappa_1 + \kappa_2)^* \beta_0}{2(1 + \kappa_1 \cdot \kappa_2)} = -g_1 g_2 \left(\frac{*\kappa_1 \kappa_2}{1 + \kappa_1 \cdot \kappa_2}\right). \tag{3.25}$$

However, as shown in Appendix B, this means

$$d\alpha_0 = \beta_1. \tag{3.26}$$

Therefore, (3.24b) together with the condition $\kappa^* \alpha_1 = 0$ yields

$$\alpha_1 \equiv 0. \tag{3.27}$$

Furthermore, (3.26) is the condition required for the exactness of the APL solution. Consequently, where $\kappa \neq 0$, *the differential of the one-form*

$$\alpha_{12} = e^{\nu\Gamma}\alpha_0 = -u_1 u_2 \left(\frac{*\kappa_1 \kappa_2}{1 + \kappa_1 \cdot \kappa_2}\right), \tag{3.28}$$

is identical to the cross-flux two-form (3.18)

$$d\alpha_{12} \equiv \beta_{12}. \tag{3.29}$$

The one-form α_0 is a singular where $\kappa_1 = -\kappa_2$, hence, where $1 + \kappa_1 \cdot \kappa_2 = 0$. This occurs along the line segment $s_1 s_2$ connecting $\mathbf{s}_1$ and $\mathbf{s}_2$. Thus, if the line $s_1 s_2$ does not intersect $\mathfrak{D}$, the field at $\mathbf{s}_2$ is

$$\beta_{12}|\mathfrak{D} = \alpha_{12}|\Sigma. \tag{3.30}$$

On the other hand, if it does intersect $\mathfrak{D}$ and because $\partial S = -\Sigma$, the field at $\mathbf{s}_2$ is

$$\beta_{12}|\mathfrak{D} = \beta_{12}|M - \beta_{12}|S = u_1(\mathbf{s}_2) + \alpha_{12}|\Sigma, \tag{3.31}$$

where the first term follows from Green's theorem. Hence, the field at $\mathbf{s}_2$ is now composed of the geometrical optics field (the first term) in addition to the diffracted field (the second term). The expression of the latter shows that it may be considered as originating on the edge Σ of the aperture, an interpretation that agrees with the viewpoint of the geometrical theory of diffraction (GTD). Furthermore, notice that the phase function Γ is stationary (i.e, $\kappa = 0$) along the line $s_1 s_2$, where $\kappa_1 = -\kappa_2$. In fact, with the results of the following section it can be shown that the geometrical optics term is recovered with the stationary phase approach. Finally, it must be re-emphasized that (3.30) and (3.31) are *exact representations* of the Kirchhoff approximation of the field resulting from the scattering of a spherical wave through an aperture.

3. *Plane wave-spherical wave case.* Consider the plane wave $u_1(\mathbf{r})=\exp[\nu(\kappa_1|\mathbf{r})]$ and the spherical wave $u_2(\mathbf{r})=G(\mathbf{r}-\mathbf{s}_2)$. The results for this case follow immediately from the preceding cases. If $\kappa_1 \neq -\kappa_2$, the one-form

$$\alpha_{12}=e^{\nu\Gamma}\alpha_0=-u_1u_2\left(\frac{*\kappa_1\kappa_2}{1+\kappa_1\cdot\kappa_2}\right), \tag{3.32}$$

where

$$\Gamma=(\kappa_1|\mathbf{r})+r_2 \tag{3.33}$$

and

$$\alpha_0=-g_2\left(\frac{*\kappa_1\kappa_2}{1+\kappa_1\cdot\kappa_2}\right), \tag{3.34}$$

is an exact solution of the equation

$$\beta_{12}|\mathfrak{D}=\alpha_{12}|\Sigma, \tag{3.35}$$

where the cross-flux two-form

$$\beta_{12}=u_1u_2*\left[\nu(\kappa_2-\kappa_1)-\frac{\kappa_2}{r_2}\right]. \tag{3.36}$$

When $\kappa_1=-\kappa_2$, the geometrical optics field $u_1(\mathbf{s}_2)$ must be added to the diffracted field in (3.35). The case dealing with a general incident field $u_1(\mathbf{r})$ can be handled (exactly) with these results by representing u_1 as a superposition of plane waves.

4. *Asymptotic field-spherical wave case.* Consider the field $u_1(\mathbf{r})=\exp[\nu\Phi(\mathbf{r})]A(\mathbf{r})$ which asymptotically satisfies the Helmholtz equation and the spherical wave $u_2(\mathbf{r})=G(\mathbf{r}-\mathbf{s}_2)$. The cross-flux two-form truncated to an asymptotic order corresponding to that of the incident field u_1 is

$$\beta_{12}\sim\nu u_1u_2*(\kappa_2-\kappa_1). \tag{3.37}$$

The subsequent asymptotic expression

$$\beta_{12}|\mathfrak{D}\sim\alpha_{12}|\Sigma, \tag{3.38}$$

where

$$\alpha_{12}=e^{\nu\Gamma}\sum_{j=0}^{m}\nu^{-j}\alpha_j, \tag{3.39}$$

follows immediately from the APL-boundary point formalism. In particular, the phase function

$$\Gamma=\Phi+r_2, \tag{3.40}$$

and if $d\Phi=\kappa_1\neq\kappa_2$,

$$\alpha_0=K^*\beta_0=-Ag_2\left(\frac{*\kappa_1\kappa_2}{1+\kappa_1\cdot\kappa_2}\right), \tag{3.41}$$

so that

$$\alpha_j=\{-K^*d\}^j\alpha_0 \qquad (1\leq j\leq m). \tag{3.42}$$

The stationary point situation (where $d\Gamma=\kappa_1+\kappa_2=0$) is handled as shown in the following section.

The preceding results recover those given in [6]–[8]. However, the discussion has been appreciably simplified using differential forms. Moreover, it refutes the statement in [8, p. 210] that it is impossible to derive in a simple way the representation and the properties of the vector potential which corresponds to the one-form $e^{\nu\Gamma}\alpha$. In fact, in all of the above cases if $\bar{\kappa}_j=\hat{r}_j$ such that $\mathbf{r}_j=r_j\hat{r}_j$, the one-form solutions

$$e^{\nu\Gamma}\alpha=-u_1u_2\left(\frac{*\kappa_1\kappa_2}{1+\kappa_1\cdot\kappa_2}\right) \tag{3.43}$$

are simply related to the vector potentials $\mathbf{W}$ of [8]

$$\mathbf{W}=e^{\nu\Gamma}\bar{\alpha}=-u_1u_2\frac{\hat{r}_1\times\hat{r}_2}{1+\hat{r}_1\cdot\hat{r}_2}. \tag{3.44}$$

Furthermore, these results are readily extended to vector-field problems. A discussion of those problems is in preparation.

4. Stationary point contributions. The asymptotic evaluation of the integral

$$e^{\nu\Gamma}\beta|X=e^{\nu\Gamma(x)}A(x)\,dx^N|X, \tag{4.1}$$

where $x=(x_1,x_2,\cdots,x_n)$ is a point in X and the volume n-form $dx^N=dx^1\,dx^2\cdots dx^n$, is desired when a nondegenerate stationary point, x_0, is the only critical point of Γ in X and the function A is smooth over X, is zero at infinity and is integrable. A point x_0 is a stationary point of Γ if

$$d\Gamma(x_0)=0; \tag{4.2}$$

it is nondegenerate if, in addition,

$$\det\left\|\left(\partial_{x_i}\partial_{x_j}\Gamma\right)(x_0)\right\|\neq 0, \tag{4.3}$$

where $\partial_{x_j}\Gamma$ means $\partial\Gamma/\partial x_j$. Note that only the case for which β is an n-form is considered explicitly. The general case in which β is a p-form integrated over a p-domain is handled in a similar manner; the generalizations of the following results to that case will be apparent. Also, if there is more than one stationary point of Γ in X and if they are not near to one another, each stationary point can be localized with neutralizers as shown, for instance, in [4] and treated like the present case. Problems involving the coalescing of stationary points, branch points, poles, and so on will not be discussed. The situation where the domain is an n-domain $\mathfrak{D}$ rather than the whole space X will be discussed at the end of this section.

In the vicinity of x_0 the Morse lemma [13] realizes a change of variables which reduces the phase to a quadratic form with coefficients (± 1). Denote the new variables by $u=(u_1,\cdots,u_n)$. They are functions of x in the vicinity of x_0; hence, they may be expressed as $u(x,x_0)$. Furthermore, they satisfy the relation

$$\Gamma(x)-\Gamma(x_0)=\frac{1}{2}\sum_{j=1}^{n}\eta_j u_j^2(x,x_0)\equiv\frac{1}{2}\eta\cdot u^2(x,x_0), \tag{4.4}$$

where $\eta_j=\pm 1$. For each x_0 there is an associated map

$$\mu_{x_0}\colon U\to X\colon u\mapsto x=\mu_{x_0}u\colon 0\mapsto x_0, \tag{4.5}$$

which expresses x in terms of the new variable u. It will be called the Morse map. The desired quadratic form:

$$Q(u)=\frac{1}{2}\sum_{j=1}^{n}\eta_j u_j^2 \tag{4.6}$$

is defined as

$$\mu_{x_0}^*[\Gamma(x)-\Gamma(x_0)]=Q(u), \tag{4.7}$$

where $\mu_{x_0}^*$ is the pullback through the Morse map, μ_{x_0}. To complete the integral (4.1), the change of variables or pullback $\mu_{x_0}^*$ must be applied to the amplitude n-form, β. This yields

$$\mu_{x_0}^*\beta=\mu_{x_0}^*[A(x)\,dx^N]=G(u,x_0)\,du^N, \tag{4.8}$$

where

$$G(u,x_0)=\mu_{x_0}^*\left[A(x)\left(\frac{du^N}{dx^N}\right)^{-1}\right]=A(\mu_{x_0}u)J(u,x_0) \tag{4.9}$$

and the Jacobian of the transformation from the u to the $x=\mu_{x_0}u$ coordinates:

$$J(u,x_0)=\left\{\mu_{x_0}^*\left[\det\left\|\partial_{x_j}u_i(x,x_0)\right\|\right]\right\}^{-1}=\det\left\|\partial_{u_j}(x=\mu_{x_0}u)_i\right\|. \tag{4.10}$$

Finally, with this change of variables the integral (4.1) becomes

$$e^{\nu\Gamma}\beta|X=e^{\nu\Gamma(x_0)}[e^{\nu Q(u)}G(u,x_0)\,du^N|U]\equiv I(\nu,x_0). \tag{4.11}$$

This expression can be rewritten immediately as

$$\begin{aligned} I(\nu,x_0)=&\,e^{\nu\Gamma(x_0)}G(0,x_0)[e^{\nu Q(u)}\,du^N|U]\\ &+e^{\nu\Gamma(x_0)}\{e^{\nu Q(u)}[G(u,x_0)-G(0,x_0)]\,du^N|U\}. \end{aligned} \tag{4.12}$$

The first integral in (4.12) is simply [4], [14]

$$e^{\nu Q(u)}\,du^N|U=c_n c^{-1}=\left(\frac{k}{\pi}\right)^{-n/2}\exp\left[i\left(\frac{\pi}{4}\right)\operatorname{sgn}\eta\right], \tag{4.13}$$

where

$$c_n=2^{-n/2}\exp\left[i\left(\frac{\pi}{2}\right)\operatorname{ind}\eta\right] \tag{4.14}$$

and

$$c=\left(\frac{k}{2\pi}\right)^{n/2}\exp\left[-\frac{in\pi}{4}\right] \tag{4.15}$$

and where if n_+ and n_- are, respectively, the number of positive and negative η_j $(j=1,2,\cdots,n)$, then the signature $\operatorname{sgn}\eta=n_+-n_-$ and the index $\operatorname{Ind}\eta=n_-$. The results of the lemmas proved in Appendix C allow one to manipulate the second integral into a form suitable for an asymptotic evaluation. In particular, let the one-form

$$\rho=\sum_{i=1}^{n}\eta_i u_i\,du^i. \tag{4.16}$$

As shown in Appendix C, the amplitude n-form $G(u,x_0)\,du^N$ can be expressed in terms of its value at the stationary point $u=0$ of $Q(u)$ and a term linear in ρ

$$G(u,x_0)\,du^N = G(0,x_0)\,du^N + \rho H_0(u,x_0). \tag{4.17}$$

A suitable choice for the $(n-1)$-form H_0 is generated with the inversion algorithm introduced in §2

$$H_0 = (\rho\cdot\rho)^{-1}\rho^*\{[G(u,x_0)-G(0,x_0)]\,du^N\}. \tag{4.18}$$

Comments on the nonuniqueness of this choice also follow from those given in §2. With (4.13) and (4.17) the expression (4.12) becomes

$$I(\nu,x_0) = (c_n c^{-1})e^{\nu\Gamma(x_0)}G(0,x_0) + e^{\nu\Gamma(x_0)}[e^{\nu Q(u)}\rho H_0(u,x_0)|U]. \tag{4.19}$$

The integral in (4.19) can be evaluated using the APL algorithm. Since

$$\beta_0 = \rho H_0 \tag{4.20}$$

is an n-form and since

$$\kappa = d_u Q = \rho, \tag{4.21}$$

the condition

$$\kappa\beta_0 = 0 \tag{4.22}$$

is trivially satisfied. Although the (sufficient) condition $\kappa\neq 0$ was needed in the preceding section, we can still give meaning to the expression $\alpha_0 = K^*\beta_0$. Because β_0, as well as κ, is zero at the stationary point $u=0$, the combination $K^*\beta_0$ has a finite limit there. Therefore, the $(n-1)$-form α_0 is well defined over U. Furthermore, since $\rho^* H_0 = 0$,

$$\alpha_0 = K^*\beta_0 = (\rho\cdot\rho)^{-1}\rho^*(\rho H_0) \equiv H_0(u,x_0). \tag{4.23}$$

With the relation

$$D_u[e^{\nu Q(u)}\alpha_0(u)] = e^{\nu Q(u)}\beta_0(u) + e^{\nu Q(u)}D_u\alpha_0, \tag{4.24}$$

one immediately obtains

$$e^{\nu Q(u)}\beta_0|U = \nu^{-1}[e^{\nu Q(u)}H_0|\partial U] + \nu^{-1}[e^{\nu Q(u)}G_1(u,x_0)\,du^N|U], \tag{4.25}$$

where

$$G_1(u,x_0)\,du^N = -d_u H_0. \tag{4.26}$$

Because the boundary, ∂U, of U is at infinity and the amplitude function A is zero there, the boundary integral in (4.25) is zero. The other integral has the same form as the original integral in (4.19); hence, this process can be repeated. After $(m+1)$ steps (4.12) becomes

$$\begin{aligned} I(\nu,x_0) &= e^{\nu\Gamma(x_0)}(c_n c^{-1})\sum_{j=0}^{m}\nu^{-j}G_j(0,x_0) \\ &\quad + \nu^{-(m+1)}e^{\nu\Gamma(x_0)}[e^{\nu Q(u)}G_{m+1}(u,x_0)\,du^N|U], \end{aligned} \tag{4.27}$$

where $G_0 = G$,

$$G_j(u,x_0)\,du^N = G_j(0,x_0)\,du^N + \rho H_j(u,x_0) \qquad (0 \le j \le m) \tag{4.28}$$

and

$$G_i(u,x_0)\,du^N = -d_u H_{i-1} \qquad (1 \le i \le m+1). \tag{4.29}$$

Truncating the remainder term in (4.27) which is of degree $(m+1)$ in ν^{-1}, the $(m+1)$-term stationary phase approximation of the integral (4.1) is

$$I(\nu,x_0) \sim e^{\nu\Gamma(x_0)}(c_n c^{-1}) \sum_{j=0}^{m} \nu^{-j} G_j(0,x_0). \tag{4.30}$$

Furthermore, as shown in Appendix D,

$$G_j(0,x_0) = \left(-\frac{1}{2}\right)^j \left\{\frac{1}{j!}\tilde{\Delta}^j_{u/0}\right\} G(u,x_0), \tag{4.31}$$

where the (modified) Laplacian operator

$$\tilde{\Delta}_u \equiv \sum_{j=1}^{n} \eta_j \partial^2_{u_j} \equiv \eta \cdot \partial^2_u \tag{4.32}$$

and

$$\tilde{\Delta}_{u/0} f(u) = \tilde{\Delta}_u f(u)|_{u=0}. \tag{4.33}$$

The standard stationary phase expressions are readily obtained from (4.30) and (4.31) [14]. Furthermore, the preceding derivation is a generalization of the ones given in [4] and [5] which employed vector identities (divergence theorem) and, hence, were restricted to Euclidean spaces. In contrast with [4] and [5], for example, the results of this section remain valid in cases where X is a manifold.

One can rewrite the expression (4.31) in a more manifest form [14]

$$cI(\nu,x_0) \sim e^{\nu\Gamma(x_0)} \sum_{j=0}^{m} \nu^{-j} T_j^{\Gamma} A(x_0), \tag{4.34}$$

where

$$T_j^{\Gamma} A(x_0) = \left\{ c_n \left(-\frac{1}{2}\right)^j \left[\frac{1}{j!}\tilde{\Delta}^j_{u/0}\right] J(u,x_0) \mu^*_{x_0} \right\} A(x) \equiv c_n G_j(0,x_0), \tag{4.35}$$

each term in the braces being treated as operators and applied in order from right to left. The advantages are that the leading term of the series is independent of k and that the contributions to the expansion from the phase and the amplitude are apparent. The operators T_j^{Γ} depend on derivatives of the phase evaluated at the stationary point at most to the order $2(j+1)$ and contain derivatives which are applied to the amplitude and evaluated at the stationary point at most to the order $2j$. For instance, when $n=1$, if $\Gamma_i = \partial^i_{x/x_0}\Gamma$, then

$$T_0^{\Gamma} = \Gamma_2^{-1/2} \partial_{x/x_0}, \tag{4.36}$$

and

$$T_1^{\Gamma} = -\frac{1}{24\Gamma_2^{3/2}} \left[(5\Gamma_3^2 - 3\Gamma_4\Gamma_2)\partial^0_{x/x_0} - 12\Gamma_3\Gamma_2\partial^1_{x/x_0} + 12\Gamma_2^2\partial^2_{x/x_0} \right]. \tag{4.37}$$

Applications and ramifications of these results are given in [14] in connection with the asymptotic evaluation of the Fourier transform.

Now consider the integral $ce^{\nu\Gamma}\beta|\mathfrak{D}$, where $\mathfrak{D}$ is an n-domain in X. Let Γ have an isolated, nondegenerate stationary point, x_0, in the interior of $\mathfrak{D}$; in particular, $d\Gamma\neq 0$ on $\Sigma=\partial\mathfrak{D}$. Let $\overline{\mathfrak{D}}$ be the complement of $\mathfrak{D}$ in X whose boundary is oriented such that $\partial\overline{\mathfrak{D}}=-\Sigma$. Thus,

$$ce^{\nu\Gamma}\beta|\mathfrak{D}\equiv ce^{\nu\Gamma}\beta|X-ce^{\nu\Gamma}\beta|\overline{\mathfrak{D}}. \tag{4.38}$$

The asymptotic evaluation of the first integral clearly yields the stationary phase contribution, $cI(\nu,x_0)\equiv F(\nu,x_0)$. Since $\overline{\mathfrak{D}}$ is devoid of any stationary points, the second integral can be handled with the boundary point formalism. Thus, with the APL solution

$$\alpha=\nu^{-1}\sum_{j=0}^{m}\nu^{-j}\alpha_j\equiv\nu^{-1}\tilde{\alpha}, \tag{4.39}$$

it gives

$$-ce^{\nu\Gamma}\beta|\overline{\mathfrak{D}}=ce^{\nu\Gamma}\alpha|\Sigma=(\nu^{-1}c)\left[e^{\nu\Gamma}\tilde{\alpha}|\Sigma\right]=\nu^{-1/2}\left[\tilde{c}e^{\nu\Gamma}\tilde{\alpha}|\Sigma\right]\equiv\nu^{-1/2}E(\nu,\Sigma), \tag{4.40}$$

where

$$\tilde{c}=-i(2\pi)^{-1/2}\left\{\left(\frac{k}{2\pi}\right)^{(n-1)/2}\exp\left[-i\left(\frac{\pi}{4}\right)(n-1)\right]\right\} \tag{4.41}$$

Therefore,

$$ce^{\nu\Gamma}\beta|\mathfrak{D}\sim F(\nu,x_0)+\nu^{-1/2}E(\nu,\Sigma); \tag{4.42}$$

the total asymptotic expansion of the integral is defined in terms of the stationary and the boundary point contributions. This result assumes those contributions are independent. Modifications of (4.42) would be necessary if this condition was not satisfied.

Notice that the boundary integral can be reparameterized in terms of local coordinates on Σ. The resulting integral can then be evaluated asymptotically. If the resultant phase function is stationary at some point (a critical point of the second kind; the stationary point x_0 in the interior of $\mathfrak{D}$ being a critical point of the first kind), the stationary phase formalism can be applied directly to that integral. In electromagnetics these terms account for the diffracted ray contributions to the total field—those rays generated from boundaries such as edges. The stationary points of the first kind, on the other hand, produce the geometrical optics terms. The characteristic $k^{-1/2}$ difference between these contributions is apparent in (4.42). However, in some instances every point on the boundary is a stationary point, and it becomes necessary to keep the nonlocal integral representation of the boundary point contributions. Similarly, if the boundary Σ has critical points such as discontinuities in its tangents (critical points of the third kind) or in its curvature (critical points of the fourth kind), the boundary Σ can be subdivided into regions over which the derivatives are continuous to a certain order and whose boundaries coincide with the points of discontinuity. The integrals over these subregions can now be treated with the boundary point formalism. Standard results given in [4] or [5] are readily recovered. Note, however, that the differential form expressions are especially suited to calculations of these types.

Finally, consider the case where x_0 is a nondegenerate stationary point of Γ and lies on the boundary Σ. Near x_0 a local coordinate system $u=(u',u_n)$ (where $u'=(u_1,\cdots,u_{n-1})$ defines a point on Σ and u_n is defined along the unit normal to Σ at x_0) can be constructed such that the transformed phase function, $\mu_{x_0}^*\Gamma$, is stationary at $u=(u',u_n)=0$ and takes the form $(\mu_{x_0}^*\Gamma)(u)=\Gamma'(u',x_0)+\eta u_n^2$, where $\eta=+1$ (-1) if u_n is positive (negative) for a point in the interior of $\mathfrak{D}$. Consequently, one has

$$e^{\nu\Gamma}\beta|\mathfrak{D}=\int_0^{\tau} e^{\nu\eta u_n^2} g(u_n,x_0;\nu)\, d(\eta u_n), \tag{4.43}$$

where

$$g(u_n,x_0;\nu)=e^{\nu\Gamma'(u',x_0)}G(u',u_n,x_0)\, du'|\Sigma \tag{4.44}$$

and where τ is a real positive constant. Note that with k large, the constant τ can be replaced with ∞, [4]. The asymptotic approximation of the original integral is obtained by applying the stationary phase approach first to the $(n-1)$-dimensional integral over Σ treating the u_n-variable as a parameter and then to the one-dimensional integral over u_n. The final result differs from (4.34) because of the form of the latter integration. Because that integration is only over the nonnegative reals, odd orders of derivatives and (referring to (4.13)) the coefficient $\frac{1}{2}$, characteristic of this case, appear. The expressions given in [4] and [5] are readily reproduced.

5. Conclusions. Exterior differential calculus techniques were used to formulate and to obtain asymptotic solutions of Poincaré's lemma. In particular, a new method of solution of a general type of differential form equation was developed. Several applications of these asymptotic Poincaré lemma results were presented. The boundary and stationary point contributions to the asymptotic approximation of a multidimensional integral were derived. Other critical point contributions and asymptotic techniques were also discussed. The boundary point approach was applied to the Kirchhoff representation of the diffraction of a scalar field through an aperture. A representation of the Leray form was synthesized that did not require the introduction of any local coordinate system. In all of these applications the resultant differential form expressions encompass, as special cases, standard vector calculus representations. Furthermore, in contrast with their vector counterparts the differential form expressions are easier to obtain and their properties are more transparent. The asymptotic Poincaré lemma and the associated techniques constitute a versatile approach to a large class of problems encountered in physics and engineering.

Appendix A: The Leray form. Consider an $(n-1)$-dimensional hypersurface S in X. A neighborhood V of a point on S can be defined by the equation $P(x_1,\cdots,x_n)=0$, where P is an infinitely differentiable function such that $dP\neq 0$ on V (i.e., there are no singular points on V). A form ω, which satisfies

$$dx^N=dP\omega, \tag{A.1}$$

is readily obtained with the inversion algorithm introduced in §2. In particular, since $dP\,dx^N\equiv 0$ (the volume form dx^N is an n-form), a solution of (A.1) is

$$\omega=\frac{dP^*\,dx^N}{dP^*\,dP}=(-1)^s\frac{*dP}{dP\cdot dP}, \tag{A.2}$$

where s is the index of the metric associated with the space X. This $(n-1)$-form is called the *Leray form* [9, Chap. III, §1]. Note that we have taken

$$*\,dx^N=1 \tag{A.3}$$

and

$$*^{-1}=(-1)^s * w^{(n+1)}, \tag{A.4}$$

where w is the operator which applied to a p-form β gives $w\beta=(-1)^p\beta$. Clearly, the Leray form depends only on the function P by which V is represented. If $P(x)$ represents (up to higher order terms) the distance from x to V, the $(n-1)$-form ω reduces to the Euclidean element of area on S. Statements concerning uniqueness follow directly from those discussed in the APL. In particular, the form $\omega+(dP)\gamma$, where γ is any $(n-2)$-form, is also a solution of (A.1). Notice that if one assumes on V some $\partial_{x_j}P\neq 0$ such that $dP=(\partial_{x_j}P)\,dx^j$, (A.2) reduces to

$$\omega=(-1)^{j-1}\frac{dx^1\cdots dx^{j-1}\,dx^{j+1}\cdots dx^n}{\partial_{x_j}P}. \tag{A.5}$$

Similar arguments can be applied to an $(n-j)$-dimensional manifold defined locally by the equations: $P_1(x)=0$, $P_2(x)=0,\cdots,P_j(x)=0$. The $(n-j)$-form

$$\omega=\frac{(dP_1\,dP_2\cdots dP_j)^*\,dx^N}{(dP_1\cdots dP_j)^*(dP_1\cdots dP_j)}=\frac{(-1)^s*(dP_1\cdots dP_j)}{(dP_1\cdots dP_j)\cdot(dP_1\cdots dP_j)} \tag{A.6}$$

satisfies the relation

$$dx^N=dP_1\,dP_2\cdots dP_j\,\omega. \tag{A.7}$$

If γ is any $(n-2j)$-form, the form $\omega+(dP_1\cdots dP_j)\gamma$ is also a solution of (A.7). Furthermore, with the expression

$$dP_1\cdots dP_j=J(P_1,\cdots,P_j;x_1,\cdots,x_j)\,dx^1\cdots dx^j, \tag{A.8}$$

where the Jacobian

$$J(P_1,\cdots,P_j;x_1,\cdots,x_j)=\det\|\partial_{x_i}P_k\|_{(i,k=1,\cdots,j)}, \tag{A.9}$$

and with the adjoint operator identity $(\alpha_1\alpha_2)^*=\alpha_2^*\alpha_1^*$, the Leray form (A.6) becomes

$$\omega=\frac{dx^{j+1}\cdots dx^n}{J(P_1,\cdots,P_j;x_1,\cdots,x_j)}. \tag{A.10}$$

The preceding results coincide with those given in [9]. Note, however, that the present approach differs from the standard construct employed in [9]. For instance, (A.5) can be derived by introducing the local coordinates $(u_1,\cdots,u_n)=u$ such that $u_i=x_i$ for $i\neq j$ and $u_j=P_j$, hence, $J(x;u)=(\partial_{x_j}P)^{-1}$ and

$$dx^N=J(x;u)\,du^1\cdots du^{j-1}\,dP\,du^{j+1}\cdots du^n. \tag{A.11}$$

Equation (A.3) is recovered immediately from (A.1) and (A.11). On the other hand, the Leray form expressions (A.2) and (A.6) are globally valid and avoid the interjection of the local coordinate system.

Appendix B: The APL solution to the Kirchhoff diffraction of two spherical waves is exact. It will be shown that the 1-term APL solution to the Kirchhoff diffraction of two spherical waves is exact; i.e., that $d\alpha_0=\beta_1$. Let $\rho_j=r_j\kappa_j$. The one-form (3.25) can then be rewritten as

$$\alpha_0=-g_1g_2\frac{*\rho_1\rho_2}{r_1r_2+\rho_1\cdot\rho_2}. \tag{B.1}$$

Let

$$A=r_1r_2(r_1r_2+\rho_1\cdot\rho_2) \tag{B.2}$$

and

$$B=2r_1r_2+\rho_1\cdot\rho_2. \tag{B.3}$$

Since

$$d(\rho_1\cdot\rho_2)=\rho_1+\rho_2, \tag{B.4}$$

one has

$$dA=(r_1r_2)^{-1}\left[\left(B+r_1^2\right)r_2^2\rho_1+\left(B+r_2^2\right)r_1^2\rho_2\right] \tag{B.5}$$

so that

$$2A-(dA^*\rho_2)=-(r_1r_2)^{-1}r_2^2\left[B(\rho_1\cdot\rho_2)+(r_1r_2)^2\right]=-\left(\frac{r_2}{r_1}\right)(r_1r_2+\rho_1\cdot\rho_2)^2 \tag{B.6}$$

and

$$2A-(dA^*\rho_1)=-\left(\frac{r_1}{r_2}\right)(r_1r_2+\rho_1\cdot\rho_2)^2. \tag{B.7}$$

Consequently, with the identity

$$d^*(h\gamma)=hd^*\gamma-(dh)^*\gamma, \tag{B.8}$$

where h is a scalar function, γ is any q-form and the codifferential operator [1, F. 19]

$$d^*=*^{-1}\,d*(-1)^q, \tag{B.9}$$

when it is applied to a q-form, and with the relation

$$d^*(\rho_1\rho_2)=2(\rho_1-\rho_2), \tag{B.10}$$

one obtains

$$\begin{aligned}-(4\pi)^2d\alpha_0=*\left[d^*\left(A^{-1}\rho_1\rho_2\right)\right]&=A^{-2}*\left[2A(\rho_1-\rho_2)+dA^*(\rho_1\rho_2)\right]\\&=A^{-2}*\left[(2A-dA^*\rho_2)\rho_1-(2A-dA^*\rho_1)\rho_2\right]\\&=(r_1r_2)^{-1}*\left(\frac{\rho_2}{r_2^2}-\frac{\rho_1}{r_1^2}\right),\end{aligned} \tag{B.11}$$

so that

$$d\alpha_0=-g_1g_2*\left(\frac{\kappa_2}{r_2}-\frac{\kappa_1}{r_1}\right)\equiv\beta_1. \tag{B.12}$$

Appendix C: Linear representation of a p-form about a point. The inversion algorithm will be used in this appendix to generalize the following lemma to p-forms.

LEMMA C.1. *Let f be a C^∞ function in a convex neighborhood M of the point $x_0=(0,\cdots,0,x_{p+1},\cdots,x_n)$ in X. Then*

$$f(x)-f(x_0)=\sum_{j=1}^{p} x_j h_j(x) \tag{C.1}$$

for some suitable C^∞ functions h_j defined in M, with $(\partial_{x_j} f)(x_0)=h_j(x_0)$.

Proof.

$$f(x)-f(x_0)=\int_0^1 \frac{df}{dt}(tx_1,\cdots,tx_p,x_{p+1},\cdots,x_n)\,dt$$

$$=\int_0^1 \sum_{j=1}^{p} \frac{\partial f}{\partial x_j}(tx_1,\cdots,tx_p,x_{p+1},\cdots,x_n)x_j\,dt.$$

Therefore, set $h_j(x)=\int_0^1(\partial f/\partial x_j)\,(tx_1,\cdots,tx_p,x_{p+1},\cdots,x_n)\,dt$. □

Note that Lemma C.1 is a simple extension of [13, Lemma 2.1]. Now consider the following lemma.

LEMMA C.2. *In X let the p-form*

$$\beta(x)=a_J(x)\,dx^J, \tag{C.2}$$

where J is the multi-index of length p: $J=j_1 j_2\cdots j_p$, so that

$$dx^J=dx^{j_1}dx^{j_2}\cdots dx^{j_p}.$$

With the subset $\mathcal{J}=\{j_1,j_2,\cdots,j_p\}$ of the set $\{1,2,\cdots,n\}$, let the one-form

$$\theta=\sum_{j\in\mathcal{J}} f_j\,dx^j, \tag{C.3}$$

where f_j is a scalar function. Then, if not all of the f_j are zero, the p-form (C.2) *has the linear representation*

$$\beta(x)=\theta\alpha(x), \tag{C.4}$$

where α is some suitable $(p-1)$-form.

Proof. The relation (C.4) follows immediately from the inversion algorithm discussed in §2. In particular, the necessary condition

$$\theta\beta(x)=0 \tag{C.5}$$

is trivially satisfied. Thus, since at least one $f_j\neq 0$, hence, $\theta\neq 0$, set

$$\alpha(x)=(\theta^*\theta)^{-1}[\theta^*\beta(x)]. \tag{C.6}$$

□

COROLLARY. *Let the one-form*

$$\theta=\sum_{j\in\mathcal{J}} c_j x_j\,dx^j, \tag{C.7}$$

where c_j is a constant, and let $x_0=(x_{10},\cdots,x_{n0})$ be the point whose components $x_{j0}=0$ for $j\in\mathcal{J}$. Then about x_0 one has the linear representation

$$\beta(x)-\beta(x_0)=\theta H(x). \tag{C.8}$$

for some suitable $(p-1)$-form H.

The corollary is clearly a special case of Lemma C.2; a suitable H is

$$H(x)=(\theta^*\theta)^{-1}\theta^*[\beta(x)-\beta(x_0)]. \tag{C.9}$$

Equation (C.8) is the desired generalization of (C.1).

Appendix D: A representation of the stationary phase amplitudes. The (modified) Laplacian

$$\tilde{\Delta}_u=\eta\cdot\partial_u^2\equiv L_u$$

applied to a p-form

$$\alpha=\sum_J a_J\, du^J,$$

where J is a multi-index of length p, acts only on its coefficients:

$$L_u\alpha=\sum_J (L_u a_J)\, du^J.$$

If the q-form

$$H_j=\sum_I h_{jI}\, du^I,$$

where I is a multi-index of length q and the one-form

$$\rho=\sum_{i=1}^n \eta_i u_i\, du^i$$

then

$$\begin{aligned} L_u(\rho H_j)&=\sum_{i=1}^n\sum_I\left[\eta_i L_u(u_i h_{jI})\right] du^i\, du^I \\ &=\sum_{i=1}^n\sum_I\left[\eta_i u_i(L_u h_{jI})\right] du^i\, du^I+2\sum_{i=1}^n\sum_{l=1}^n\sum_I\left(\eta_i\eta_l\partial_{u_l}u_i\partial_{u_l}h_{jI}\right) du^i\, du^I \\ &=\rho L_u H_j+2\sum_{i=1}^n\sum_I\left(\partial_{u_i}h_{jI}\right) du^i\, du^I=\rho L_u H_j+2d_u H_j. \end{aligned} \tag{D.1}$$

With the identity

$$d_u\circ L_u=L_u\circ d_u,$$

repeated applications of (D.1) give

$$\begin{aligned} L_u^m(\rho H_j)&=L_u^{m-1}(\rho L_u H_j+2d_u H_j)=L_u^{m-2}\left[\rho L_u^2 H_j+2d_u(L_u H_j)+2L_u(d_u H_j)\right] \\ &=L_u^{m-2}\left[\rho L_u^2 H_j+4L_u(d_u H_j)\right]=\cdots=\rho L_u^m H_j+2mL_u^{m-1}(d_u H_j). \end{aligned}$$

Thus,

$$\tilde{\Delta}_{u/0}^m(\rho H_j)=2m\tilde{\Delta}_{u/0}^{m-1}(d_u H_j). \tag{D.2}$$

Now consider the p-form $\beta(u,x_0)$ and the $(p-1)$-form $H_j(u,x_0)$ which satisfy the system

$$\beta_j(u,x_0)=\beta_j(0,x_0)+\rho H_j(u,x_0) \qquad (0\leq j\leq m),$$
$$\beta_j(u,x_0)=-d_u H_{i-1} \qquad (1\leq i\leq m+1).$$

Equation (D.2) then gives

$$\tilde{\Delta}^m_{u/0}\beta_j(u,x_0)=-2m\tilde{\Delta}^{m-1}_{u/0}\beta_{j+1}(u,x_0).$$

Therefore,

$$\tilde{\Delta}^j_{u/0}\beta_0(u,x_0)=-2j\tilde{\Delta}^{j-1}_{u/0}\beta_1(u,x_0)=(-2)^2 j(j-1)\tilde{\Delta}^{j-2}_{u/0}\beta_2(u,x_0)$$
$$=\cdots=(-2)^j j!\beta_j(0,x_0),$$

hence

$$\beta_j(0,x_0)=\left(-\frac{1}{2}\right)^j\left\{\frac{1}{j!}\tilde{\Delta}^j_{u/0}\right\}\beta_0(u,x_0). \tag{D.3}$$

Consequently, with the n-form

$$\beta_j(u,x_0)=G_j(u,x_0)\,du^N,$$

(D.3) yields

$$G_j(0,x_0)=\left(-\frac{1}{2}\right)^j\left\{\frac{1}{j!}\tilde{\Delta}^j_{u/0}\right\}G_0(u,x_0). \tag{D.4}$$

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