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Weakly nonlinear geometrical optics in plasmas

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A geometrical optics method for solving wave propagation in dispersive weakly nonlinear, weakly inhomogeneous media is applied to plasmas. In the case that the dispersion is comparable to the nonlinearity, the method yields the expected soliton solutions. For strong dispersion, it is found that the amplitude has the same form as in the linear case except that it is multiplied by a slow varying sinusoidal part.

1. Introduction

The problem of wave propagation in nonlinear media is of long and continuing interest. For intense beams various nonlinear phenomena such as self-focusing [1] can occur. In plasmas intense electromagnetic waves can drive the system nonlinear and can cause, among several effects, the parametric interaction between the various modes [2], or the formation of solitons [3]. For fusion devices, the electric field strength needed to reach the threshold for parametric interaction is on the order of a few tens of kilovolts per meter [4]. Such field strength is typical for heating plasmas to fusion temperatures. Thresholds for soliton formation are also attainable. Therefore, in order to understand the heating of a plasma with microwaves, it is essential to study nonlinear wave propagation in plasmas.

In this paper we apply a geometrical optics method to the propagation of waves in a weakly nonlinear, weakly inhomogeneous, and dispersive plasma medium. We concentrate here on single wave propagation. We have previously treated the problem of the propagation of a Gaussian beam in a nondispersive nonlinear medium [5]. Our approach makes use of a perturbation method devised by Choquet–Bruhat [6,7]. The solution is expressed as an asymptotic expansion in terms of a small parameter δ which is also a measure of the period of the wave. The problem is then reduced to a set of decoupled equations that describes the rays and the amplitude of the wave. For weak dispersion, the zeroth order equation yields nondispersive linear rays. The amplitude, however, is described by a nonlinear equation that admits soliton solutions. For strong dispersion, the zeroth order equation provides the dispersion relation. The first order equation provides an amplitude that consists of a part that is similar to the linear solution, multiplied by a slow varying sinusoidal part. Other methods such as Whitham's variational technique require two coupled equations for the phase and the amplitude [8].

The plan of the paper is as follows: In section 2 we will derive the ray and transport equation for the weakly dispersive case, where the dispersion and the nonlinearity are comparable, and show that it produces the well-known soliton solutions. We confine our analysis to scalar variables in one spatial dimension. In section 3 we will derive the ray and transport equation for the strongly dispersive case, where the dispersion is much greater than the nonlinearity. The analysis for section 3 is generalized to vector fields in multidimensions. We summarize our results in section 4.

2. Case of weak dispersion

The geometrical optics method is now applied to the weak dispersion case where the dispersive and nonlinear terms are assumed to be of the same order. We consider here the case of ion acoustic wave propagating in an inhomogeneous unmagnetized plasma medium. The dispersion is characterized by assuming that the square of the Debye length is of the order δ^3. The ion distribution function f is expressed as

$$f = f_0(x, t, v) + \sum_{n=1}^{n=\infty} \delta^n f_n(x, t, \phi/\delta, v) , \tag{1}$$

where f_0 is the background distribution function and ϕ is a phase function. The ion distribution function satisfies the Vlasov equation:

$$\partial_t f + v\, \partial_x f - \frac{e}{m_i} (\partial_x \Phi)\, \partial_v f = 0 , \tag{2}$$

where Φ is the electrostatic potential and is expressed as

$$\Phi = \Phi_0(x, t) + \sum_{n=1}^{\infty} \delta^n \Phi_n(x, t, \phi/\delta) . \tag{3}$$

The ion density n_i is related to f by

$$n_i = \int f \,\mathrm{d}v , \tag{4}$$

The zeroth order (background) distribution function and potential satisfy the zeroth order Vlasov and Poisson equation:

$$\partial_t f_0 + v\, \partial_x f_0 - \frac{e}{m_i} (\partial_x \Phi_0)\, \partial_v f_0 = 0 , \tag{5}$$

$$\partial_x^2 \Phi_0 = \frac{e}{\epsilon_0} (n_{0e} - n_0) , \tag{6}$$

where n_{0e} and n_0 are the background electron and ion density, respectively, and m_i is the ion mass. We assume that the electron density is described by the Boltzmann distribution:

$$n_e = n_0 \exp(e\Phi/KT) = n_0\left(1 + \frac{e\Phi}{KT} + \frac{e^2\Phi^2}{2(KT)^2} + \cdots\right) , \tag{7}$$

where T is the electron temperature.

We now use eqs. (1) and (3) in eq. (2). The operators ∂_t and ∂_x are replaced by $\partial_t + \delta^{-1}(\partial_t\phi)\,\partial_\theta$ and $\partial_x + \delta^{-1}(\partial_x\phi)\,\partial_\theta$, respectively, where $\theta = \phi/\delta$ is a rapidly varying phase function.

Equating terms of order δ^0 in the Vlasov equation, one obtains:

$$-\omega\,\partial_\theta f_1 + kv\,\partial_\theta f_1 - \frac{e}{m_i}\,k(\partial_\theta \Phi_1)\,\partial_v f_n = 0\,, \tag{8}$$

where we define $\omega = -\partial_t\theta$ and $k = \partial_x\theta$. The first order distribution function is then given by

$$\partial_\theta f_1 = \frac{e}{m_i}\,\frac{k(\partial_\theta \Phi_1)\,\partial_v f_0}{kv - \omega}\,. \tag{9}$$

We now apply the operator ∂_θ to the Poisson equation and equate the terms of order δ^2. We obtain the following equation:

$$\frac{1}{\lambda_{De}^2}\,\partial_\theta \Phi_1 - \frac{e(\partial_\theta \Phi_1)}{m_i}\int \frac{\partial_v f_0}{(v - \omega/k)}\,\mathrm{d}v = 0\,, \tag{10}$$

where $\lambda_{De}^2 = \epsilon_0 KT/e^2 n_0$ is the square of the Debye length, and where we use $\lambda_{De}^2 \approx \mathcal{O}(\delta^3)$. This equation has nontrivial solutions iff

$$\frac{1}{\lambda_{De}^2} - \frac{e}{m_i}\int \frac{\partial_v f_0}{v - \omega/k}\,\mathrm{d}v = 0\,. \tag{11}$$

This equation is the eikonal equation that describes the rays along which the amplitude propagates.

For cold ions with a drift velocity V, the background ion distribution function can be expressed as

$$f_0 = n_0(x,t)\,\delta(v - V(x,t))\,, \tag{12}$$

In this case the eikonal equation gives

$$\omega/k = c_s + V(x,t)\,, \tag{13}$$

$$\mathrm{d}\phi/\mathrm{d}s = 0\,, \tag{14}$$

$$\mathrm{d}t/\mathrm{d}s = 1\,, \tag{15}$$

$$\mathrm{d}x/\mathrm{d}s = c_s + V(x,t)\,, \tag{16}$$

where s is an arclength along the ray, and $c_s = (KT/m_i)^{1/2}$ is the ion sound speed.

Equating terms of order δ in the Vlasov equation, one obtains the following expression for $\partial_\theta f_2$:

$$\partial_\theta f_2 = \frac{1}{\delta(kv - \omega)}\Big(\partial_t f_1 + v\,\partial_x f_1 - \frac{e}{m_i}(\partial_x \Phi_1)\,\partial_v f_0 - \frac{e(\partial_x\phi)}{m_i}(\partial_\theta \Phi_1)\,\partial_v f_1 - \frac{e}{m_i}(\partial_\theta \Phi_2)\,\partial_v f_0$$
$$- \frac{e}{m_i}(\partial_x \Phi_0)\,\partial_v f_1\Big) = 0\,. \tag{17}$$

We apply the operator ∂_θ to the corresponding Poisson and use the expressions for f_1 and f_2 given in equations (9) and (17) respectively, to obtain

$$k^2\,\partial_\theta^3\Phi_1 = \frac{e\delta}{KT\lambda_{\mathrm{De}}^2}\,\Phi_1\,\partial_\theta\Phi_1 + \frac{e^2}{\varepsilon_0 m_i k}\Bigg[(\partial_t\Phi_1)\int\frac{\partial_v f_0}{(v-\omega/k)^2}\,\mathrm{d}v + \Phi_1\int\frac{1}{v-\omega/k}\,\partial_t\left(\frac{\partial_v f_0}{v-\omega/k}\right)\mathrm{d}v$$
$$+ (\partial_x\Phi_1)\int\frac{v\,\partial_v f_0}{(v-\omega/k)^2}\,\mathrm{d}v + \Phi_1\int\frac{v}{v-\omega/k}\,\partial_x\left(\frac{\partial_v f_0}{v-\omega/k}\right)\mathrm{d}v - (\partial_x\Phi_1)\int\frac{\partial_v f_0}{(v-\omega/k)}\,\mathrm{d}v$$
$$-\frac{e(\partial_x\phi)}{m_i}\,(\partial_\theta\Phi_1)\Phi_1\int\frac{1}{v-\omega/k}\,\partial_v\left(\frac{\partial_v f_0}{v-\omega/k}\right)\mathrm{d}v - \frac{e}{m_i}\,(\partial_x\Phi_0)(\partial_\theta\Phi_1)\int\frac{1}{v-\omega/k}$$
$$\times\,\partial_v\left(\frac{\partial_v f_0}{v-\omega/k}\right)\mathrm{d}v\Bigg]\,, \tag{18}$$

where the terms containing Φ_2 vanish by virtue of the eikonal equation. Note that for a homogeneous medium this equation reduces to the KdV equation obtained by the reductive perturbation method [9,10].

We examine the case of an ion distribution function of the form given in eq. (12). Performing the integration in eq. (18) is straightforward. We illustrate here the integration of the fifth term on the right hand side of eq. (18):

$$\int\frac{v}{v-\omega/k}\,\partial_x\left[\frac{\partial_v f_0}{v-\omega/k}\right]\mathrm{d}v = (\partial_x n_0)\int\frac{v}{(v-\omega/k)^2}\,\partial_v(\delta[v-V(x,t)])\,\mathrm{d}v$$
$$+\,n_0\,\partial_x(\omega/k)\int\frac{v}{(v-\omega/k)^3}\,\partial_v(\delta[v-V(x,t)])\,\mathrm{d}v$$
$$-\,n_0(\partial_x V)\int\frac{v}{(v-\omega/k)^2}\,\partial_v^2(\delta[v-V(x,t)])\,\mathrm{d}v$$
$$= \frac{-3n_0(\partial_x V)(c_s+V)}{c_s^4} - \frac{(\partial_x n_0)(c_s+V)}{c_s^3} - \frac{V(\partial_x n_0)}{c_s^3}\,. \tag{19}$$

Note that since c_s is constant, eq. (13) provides $\partial_x(\omega/k) = \partial_x V$. Eq. (18) then reduces to

$$k^3 c_s\lambda_{\mathrm{De}}^2\,\partial_\theta^3\Phi_1 = \frac{e(\partial_x\phi)}{c_s m_i}\,\Phi_1\,\partial_\theta\Phi_1 - 2\partial_t\Phi_1 - 2\,\frac{(\partial_t n_0)}{n_0}\,\Phi_1 - \frac{3(\partial_t V)}{c_s}\,\Phi_1 - c_s\,\partial_x\Phi_1 - 2V\,\partial_x\Phi_1$$
$$-\frac{3(\partial_x V)(c_s+V)}{c_s}\,\Phi_1 - \frac{(\partial_x n_0)}{n_0}\,(c_s+V)\Phi_1 - \frac{(\partial_x n_0)}{n_0}\,V\Phi_1 - c_s\,\partial_x\Phi_1$$
$$-\frac{3e(\partial_x\phi)}{c_s m_i}\,\Phi_1\,\partial_\theta\Phi_1 - \frac{3e}{c_s m_i}\,(\partial_x\Phi_0)\Phi_1\,. \tag{20}$$

Using the zero and first moment equations (continuity and momentum balance equations) obtained from the Vlasov equation that describes the background state [11], and defining $\Phi_1 = KTv_1/ec_s$, one can after some manipulation reduce eq. (20) to

$$\partial_t v_1 + (c_s+V)\,\partial_x v_1 + (\partial_x\phi)v_1\,\partial_\theta v_1 + \tfrac{1}{2}(c_s+V)\,\frac{(\partial_x n_0)}{n_0}\,v_1 + \frac{1}{2}\,\frac{(\partial_t n_0)}{n_0}\,v_1 + (\partial_x V)v_1 + k^3\alpha\,\partial_\theta^3 v_1 = 0\,, \tag{21}$$

where $\alpha = c_s \lambda_{De}^2/2$. Higher order equations can be obtained by carrying on similarly. It is straightforward to show that for a time-independent medium this result is similar to that obtained by Kuehl [12]. Kuehl includes the effect of ionization in his derivation. This can also be readily done in the analysis above. For ionization by thermal electrons, the transport equation for a time independent medium will be the same as eq. (21). The background state should, however, be determined by including the ionization.

The eikonal equation shows that $\partial_s = \partial t + (c_s + V)\,\partial_x$. Therefore the transport equation (21) can be expressed along a ray as

$$\partial_s v_1 + (\partial_x \phi) v_1\, \partial_\theta v_1 + k^3 \alpha\, \partial_\theta^3 v_1 + \frac{(\partial_s n_0)}{2n_0}\, v_1 + (\partial_x V) v_1 = 0\,. \tag{22}$$

This amplitude transport equation represents a family of KdV-like equations that correspond to each ray. The value of $\partial_x \phi$ along the ray can be determined by noting from the eikonal equation that $\mathrm{d}(\partial_x \phi)/\mathrm{d}t = -(\partial_x \phi)\,\partial_x V$. We will now discuss various special solutions to this equation.

In the case of constant drift velocity, eq. (22) becomes

$$\partial_s v_1 + (\partial_x \phi) v_1\, \partial_\theta v_1 + k^3 \alpha\, \partial_\theta^3 v_1 + \frac{(\partial_s n_0)}{2n_0}\, v_1 = 0\,. \tag{23}$$

Generally, the solution of this equation can be expressed as

$$v_1 = g(\phi'/\delta, s, \xi)\, G(s, \xi)\,, \tag{24}$$

where ξ is a parameter that characterizes a given ray, and ϕ' is the nonlinear phase that can depend on ϕ, s, and ξ. We first examine the case of a homogeneous medium where α is constant and the term containing $(\partial_s n_0)/n_0$ vanishes. We also assume zero drift velocity. The nonlinear phase can be expressed as

$$\phi' = \phi - \delta c_1 t\,. \tag{25}$$

The term $\delta c_1 t$, where c_1 is a constant, is a first order correction to the linear phase. The solution is expressed as

$$v_1 = g(\phi'/\delta)\,, \tag{26}$$

where g is a function that describes the wave form. Eq. (23) then gives

$$-c_1\, \partial_{\theta'} g + (\partial_x \phi) g\, \partial_{\theta'} g + k^3 \alpha\, \partial'^3_\theta g = 0\,, \tag{27}$$

where $\theta' = \phi'/\delta$. This equation can be integrated with respect to θ' yielding

$$-c_1 g + (\partial_x \phi)\, \frac{g^2}{2} + k^3 \alpha\, \partial_{\theta'}^2 g = 0\,, \tag{28}$$

where we set the integration constant to be zero. Multiplying eq. (28) by $\partial_{\theta'} g$ and then integrating with respect to θ', one obtains

$$\frac{k^3\alpha}{2}(\partial_{\theta'} g)^2 = c_1 \frac{g^2}{2} - (\partial_x \phi)\frac{g^3}{6}, \tag{29}$$

where again we set the integration constant to be zero. Eq. (29) can then be solved implicitly in the form

$$\int \frac{\mathrm{d}g}{\sqrt{g^2 - c_2 g^3}} = \left(\frac{c_1}{k^3\alpha}\right)^{1/2}\theta', \tag{30}$$

where $c_2 = (\partial_x \phi)/3c_1$. We choose here the phase initial condition that yields $\phi = x - c_s t$. This choice obviously satisfies the eikonal equation. In this case $k = \delta^{-1}$. After some standard manipulations, eq. (30) yields

$$g = 3c_1 \operatorname{sech}^2\left(\left(\frac{\delta c_1}{4\alpha}\right)^{1/2}(x - c_s t - \delta c_1 t)\right). \tag{31}$$

Expressing $c_s + \delta c_1$ as v_0, one obtains the solution

$$\delta v_1 = 3(v_0 - c_s)\operatorname{sech}^2\left(\left(\frac{v_0 - c_s}{4\alpha}\right)^{1/2}(x - v_0 t)\right). \tag{32}$$

The above equation is the well-known solution to the KdV equation [13].

We now examine eq. (23) in the case of a spatially and temporally inhomogeneous medium. The solution can be expressed as $v_1 = u/n_0^{1/2}$, where u satisfies the equation

$$\partial_s u + \frac{1}{n_0^{1/2}}\, u\, \partial_\theta u + \delta^{-3}\alpha\, \partial_\theta^3 u = 0, \tag{33}$$

where we assume a phase initial condition that yields $\phi = x - (c_s + V)t$. For sufficiently small $(\partial_s n_0/n_0)\theta$, an approximate solution to eq. (23) can be obtained. We employ the following change of variables:

$$\rho = \int_0^s n_0^{-1/4}\,\mathrm{d}s', \tag{34}$$

$$\kappa = n_0^{1/4}\theta. \tag{35}$$

Equation (33) then yields:

$$\partial_\rho u_1 + u_1\, \partial_\kappa u_1 + \delta^{-3}\alpha_1\, \partial_\kappa^3 u_1 = 0, \tag{36}$$

where $\alpha_1 = \alpha/n_0$. This equation is similar to the KdV equation that describes a homogeneous medium. It is then straightforward to obtain the following solution:

$$\delta v_1 = 3\delta\,\frac{C}{n_0^{1/2}}\operatorname{sech}^2\left[\left(\frac{\delta C}{4\alpha_1}\right)^{1/2}\left(n_0^{1/4}\theta - \delta C\int_0^s n_0^{-1/4}\,\mathrm{d}s\right)\right], \tag{37}$$

where C is a constant. The integral in eq. (37) is carried along the ray. The ray can be characterized by $\xi = x - (c_s + V)t = \text{constant}$. This solution is similar to that obtained by Nishikawa and Kaw [14] for a spatially inhomogeneous medium. The dependence of their solution on time can be understood by noting that the solution is along a given ray. It should be mentioned, however, that in a time-independent medium, the background state cannot be achieved without the inclusion of non-uniform drift and ionization [12]. We do not have to account for this in our case since we are dealing with a medium which is both spatially and temporally inhomogeneous. Additionally, we note that a solution similar to that of Gell and Gomberoff [15] can be obtained in the case of an inhomogeneity that is of the order of δ. In this case the term containing $(\partial_s n_0/n_0)$ in eq. (23) vanishes and the solution can be readily obtained.

A hypothetical case of eq. (23) that can be readily solved by the geometrical optics method is the case of a driftless plasma in which the inhomogeneity is a function of $x - c_s t$. This may be achieved with the proper ionization. Suppose that the phase at $t = 0$ is given by $F(x)$. Again, the solution to eq. (24) can be expressed as $v_1 = u/n_0^{1/2}$ where u satisfies

$$\partial_s u + \frac{(\partial_x \phi)}{n_0^{1/2}}\, u\, \partial_\theta u + k^3 \alpha\, \partial_\theta^3 u = 0\,. \tag{38}$$

Since $\xi = x - c_s t$ is constant along a given ray, this equation can be solved in a manner similar to the case of a homogeneous medium. The solution yields

$$u = 3(\partial_\xi F)^2 h(\xi) n_0^{-1/2}\, \text{sech}^2\!\left(\frac{1}{\delta}\,[\phi - \delta n_0^{-1}(\partial_\xi F)^3 ht]\right), \tag{39}$$

where $h(\xi)$ is a function that is constant along the ray and can be determined from the initial conditions.

For a time-independent velocity, the solution to the transport equation (22) can be expressed as:

$$v_1 = \frac{u}{n_0^{1/2}(c_s + v)}\,, \tag{40}$$

where u satisfies the equation:

$$\partial_s u + \frac{(\partial_x \phi)}{n_0^{1/2}}\, u\, \partial_\theta u + k^3 \alpha\, \partial_\theta^3 u = 0\,. \tag{41}$$

An approximate solution to this equation can be obtained in a manner similar to the method described for eq. (33).

It is important to note here that the solutions obtained above are not valid for times of order δ^{-1}. To extend the validity of the method, one has to assume that the ion distribution function f and the electrostatic potential Φ are of the form

$$f = f_0(x, t, v) + \sum_{n=1}^{\infty} \delta^n f_n(x, t, \phi/\delta, X, T, v)\,, \tag{42}$$

$$\Phi = \Phi_0(x, t) + \sum_{n=1}^{\infty} \delta^n \Phi_n(x, t, \phi/\delta, X, T)\,, \tag{43}$$

where $X = \delta x$ and $T = \delta t$ are slow scale variables and ϕ is a function of x, t, X, and T. In this case, eq. (41), for example, will give

$$\partial_s u + (\partial_T \phi)\, \partial_\theta u + \frac{1}{n_0^{1/2}} (\partial_X \phi) u\, \partial_\theta u + \alpha k^3\, \partial_\theta^3 u = 0\,. \tag{44}$$

If we assume that the parameters are dependent on the slow-scale variable X, then this equation can be solved in a manner similar to that of Ko and Kuehl [16] or that of Grimshaw [17].

3. Case of strong dispersion

The geometrical optics method is now applied to the strong dispersion case. We assume here that the dispersion is much larger than the nonlinearity. We begin with Maxwell's equations in an inhomogeneous nonlinear medium

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\, \partial_t \boldsymbol{E} + \mu_0\, \partial_t \boldsymbol{D}_{\text{nl}}\,, \tag{45}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \tag{46}$$

where $\boldsymbol{j}$ represents the total linear current. By "linear current" we mean that part of the current that is proportional to the electric field. The nonlinear displacement $\boldsymbol{D}_{\text{nl}}$ represents that part of the displacement that depends on higher powers of the electric field. The total linear current is given by

$$\boldsymbol{j} = \boldsymbol{j}_{\text{e}} + \boldsymbol{j}_{\text{m}}\,,$$

where $\boldsymbol{j}_{\text{e}}$ represents the electric current, $\boldsymbol{j}_{\text{m}}$ the magnetic current. Note that for an anisotropic background distribution function the magnetic current must be included [16]. In a general nonstationary, inhomogeneous, spatially and temporally dispersive medium, the electric and magnetic current are given by

$$\boldsymbol{j}_{\text{e}}(\boldsymbol{x}, t) = \int \mathrm{d}^3 x' \int_{-\infty}^{t} \mathrm{d}t'\, \sigma_{\text{e}}(\boldsymbol{x} - \boldsymbol{x}', t - t', \tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{x}')\,, \tfrac{1}{2}(t + t')) \cdot \boldsymbol{E}(\boldsymbol{x}', t')\,, \tag{47}$$

$$\boldsymbol{j}_{\text{m}}(\boldsymbol{x}, t) = \int \mathrm{d}^3 x' \int_{-\infty}^{t} \mathrm{d}t'\, \sigma_{\text{m}}(\boldsymbol{x} - \boldsymbol{x}', t - t', \tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{x}'), \tfrac{1}{2}(t + t')) \cdot \boldsymbol{B}(\boldsymbol{x}', t')\,, \tag{48}$$

where σ_{e} and σ_{m} represent the electric and magnetic conductivity tensor respectively. Note that the central-averaged representation of the current is used. This is a particular choice; other representations are possible and have been discussed in the literature [18,19].

We make here the same assumptions that are pertinent to linear geometrical optics in plasmas. In addition, the nonlinear term is assumed to be of the order δ. We model the nonlinearity to be of the form:

$$D_{\mathrm{nl},i} \propto (\boldsymbol{E}^* \cdot \boldsymbol{E}) E_i ,$$

where E^* is the complex conjugate of E. We assume here single mode interaction and a nonlinearity which is intensity-dependent or pondermotive-like in nature. We assume that since the dispersion is strong compared to the nonlinearity, a sinusoidal form of the field may be retained. The electric and magnetic fields are expressed as asymptotic expansions in terms of δ which also measures the period of the wave:

$$\boldsymbol{E}(\boldsymbol{x}, t) = [\boldsymbol{a}_0(\boldsymbol{x}, t) + \delta \boldsymbol{a}_1(\boldsymbol{x}, t)] \exp(\mathrm{i}\theta) + \mathcal{O}(\delta^2) , \tag{49}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = [\boldsymbol{b}_0(\boldsymbol{x}, t) + \delta \boldsymbol{b}_1(\boldsymbol{x}, t)] \exp(\mathrm{i}\theta) + \mathcal{O}(\delta^2) , \tag{50}$$

where θ is a fast varying phase function.

Defining $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{x}'$, and $T = t - t'$, one can express the linear electric current as:

$$\boldsymbol{j}_{\mathrm{e}}(\boldsymbol{x}, t) = \int \mathrm{d}^3X \int_0^\infty \mathrm{d}T\, \sigma_{\mathrm{e}}(\boldsymbol{X}, T, \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{X}, t - \tfrac{1}{2}T) \cdot \boldsymbol{E}(\boldsymbol{x} - \boldsymbol{X}, t - T) . \tag{51}$$

The magnetic current can be expressed similarly. The electric field at the point $(\boldsymbol{x}', t')$ is now expanded in a Taylor's series around the point $(\boldsymbol{x}, t)$:

$$\boldsymbol{E}(\boldsymbol{x}', t') = [\boldsymbol{a}(\boldsymbol{x}, t) - (\mathrm{d}_{x_i}\boldsymbol{a})X_i - (\mathrm{d}_t\boldsymbol{a})T + \tfrac{1}{2}\mathrm{i}\boldsymbol{a}(\partial_{x_i}k_j)X_iX_j - \mathrm{i}\boldsymbol{a}(\mathrm{d}_{x_i}\omega)X_iT - \tfrac{1}{2}\mathrm{i}\boldsymbol{a}(\partial_t\omega)T^2] \times \exp[\mathrm{i}\theta(\boldsymbol{x}, t)] \exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{X} - \omega T)] + \cdots , \tag{52}$$

where $\omega = -\partial_t\theta$ and $\boldsymbol{k} = \nabla\theta$. The expansion for σ around the point $(\boldsymbol{x}, t)$ is given by

$$\sigma(\boldsymbol{X}, T, \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{X}, t - \tfrac{1}{2}T) = \sigma(\boldsymbol{X}, T, \boldsymbol{x}, t) - \tfrac{1}{2}(\mathrm{d}_{x_i}\sigma)X_i - \tfrac{1}{2}(\mathrm{d}_t\sigma)T + \cdots . \tag{53}$$

Using eq. (52) and eq. (53) in eq. (51) and performing the integral, one obtains the following equation for the zeroth order linear electric current:

$$\boldsymbol{j}_0 = \tilde{\sigma}_{\mathrm{e}}^{\mathrm{A}}(\boldsymbol{k}, \omega, \boldsymbol{x}, t) \cdot \boldsymbol{a}_0 , \tag{54}$$

where $\tilde{\sigma}_{\mathrm{e}}^{\mathrm{A}}$ is the anti-Hermitian part of the Fourier transform of σ_{e}. The Fourier transform of σ_{e} is given by

$$\tilde{\sigma}_{\mathrm{e}}(\boldsymbol{k}, \omega\ \boldsymbol{x}, t) = \int \mathrm{d}^3X \int_0^\infty \sigma_{\mathrm{e}}(\boldsymbol{X}, T; \boldsymbol{x}, t) \exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{X} - \omega T)] . \tag{55}$$

We assume that the dissipation of the wave is small. In particular, the dissipative part of the conductivity tensor is represented by the Hermitian part of $\tilde{\sigma}_{\mathrm{e}}$ and is assumed to be small: $\tilde{\sigma}_{\mathrm{e}}^{\mathrm{H}}/\tilde{\sigma}_{\mathrm{e}}^{\mathrm{A}} \approx \mathcal{O}(\delta)$.

Similarly, the zeroth order linear magnetic field is given by

$$j_{0m} = \tilde{\sigma}_m^A \cdot b_0 \,, \tag{56}$$

where $\tilde{\sigma}_m^A$ represents the Hermitian part of the Fourier transform of the magnetic conductivity.

The following equations are obtained by equating terms of order δ^{-1} in Maxwell's equations:

$$\boldsymbol{k} \times \boldsymbol{a}_0 - \omega \boldsymbol{b}_0 = 0 \,, \tag{57}$$

$$\boldsymbol{k} \times \boldsymbol{b}_0 + \frac{\omega}{c^2} \boldsymbol{a}_0 = -\mathrm{i}\mu_0[\tilde{\sigma}_e^A \cdot \boldsymbol{a}_0 + \tilde{\sigma}_m^A \cdot \boldsymbol{b}_0] \,. \tag{58}$$

Solving for $\boldsymbol{b}_0$ in eq. (57) one obtains

$$\boldsymbol{b}_0 = \frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega} \,. \tag{59}$$

Eq. (58) then reduces to

$$\boldsymbol{k} \times \left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right) + \frac{\omega}{c^2} \boldsymbol{a}_0 = -\mathrm{i}\mu_0\left[\tilde{\sigma}_e^A \boldsymbol{a}_0 + \tilde{\boldsymbol{\sigma}}_m^A \cdot \left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{w}\right)\right] \,. \tag{60}$$

This equation can be written in the form

$$\boldsymbol{N}^H \cdot \boldsymbol{a}_0 = 0 \,, \tag{61}$$

where $\boldsymbol{N}^H$ is a Hermitian tensor and is expressed explicitly as

$$N_{ij}^H = (k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\,\delta_{ij} - \mathrm{i}\mu_0 \omega \tilde{\sigma}_{ij}^A)/\omega \,, \tag{62}$$

where $\tilde{\sigma}_{ij}^A$ represents the total anti-Hermitian conductivity:

$$\tilde{\boldsymbol{\sigma}}^A = \tilde{\boldsymbol{\sigma}}_e^A + \frac{\boldsymbol{\sigma}_m^A \times \boldsymbol{k}}{\omega} \,.$$

Eq. (61) has a nontrivial solution for $\boldsymbol{a}_0$ if

$$N = \det \boldsymbol{N}^H = 0 \,. \tag{63}$$

This equation is the same as the dispersion relation of linear geometrical optics. It provides the allowed values of $(\boldsymbol{k}, \omega)$ at each point $(\boldsymbol{x}, t)$. Since $\boldsymbol{k}$ and ω are first derivatives of the linear phase, it is a first order partial differential equation. The solution curves of the first order equations set determined by the variation of N with respect to the phase space coordinates $(\boldsymbol{x}, t, \boldsymbol{k}, \omega)$ describe the rays:

$$\mathrm{d}x_i/\mathrm{d}\tau = \partial_{k_i} N \,, \tag{64}$$

$$\mathrm{d}t/\mathrm{d}\tau = -\partial_\omega N \,, \tag{65}$$

$$\mathrm{d}k_i/\mathrm{d}\tau = -\partial_{x_i} N \,, \tag{66}$$

$$\mathrm{d}\omega/\mathrm{d}\tau = \partial_t N \,. \tag{67}$$

The linear phase function θ is then given by

$$\theta(\tau) = \theta(0) + \int_0^\tau \left(\boldsymbol{k} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} - \omega \, \frac{\mathrm{d}t}{\mathrm{d}\tau} \right) \mathrm{d}\tau \,. \tag{68}$$

The following identities are now used to determine the first order current:

$$\mathrm{i}\boldsymbol{\nabla}_k \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}) = \boldsymbol{X}\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}) \,, \quad -\mathrm{i}\partial_\omega \exp(\mathrm{i}\omega T) = T\exp(\mathrm{i}\omega T) \,.$$

The first order terms in eq. (51) then yield an expression for the first order linear electric current:

$$\boldsymbol{j}_{1\mathrm{e}} = \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{H}} \cdot \boldsymbol{a}_0 + (\partial^\nu \boldsymbol{\sigma}_{\mathrm{e}}^{\mathrm{A}}) \cdot \mathrm{d}_\nu \boldsymbol{a}_0 - \tfrac{1}{2} \mathrm{d}_\nu (\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{A}}) \cdot \boldsymbol{a}_0 + \boldsymbol{\sigma}_{\mathrm{e}}^{\mathrm{A}} \cdot \boldsymbol{a}_1 \,. \tag{69}$$

where we use the four-vector notation

$$\mathrm{d}_\nu = (\mathrm{d}_x, \mathrm{d}_{\mathrm{i}t}) \,,$$

$$\partial^\nu = (\partial_k, \partial_{\mathrm{i}\omega}) \,.$$

Similarly, the first order magnetic current is given by

$$\boldsymbol{j}_{1\mathrm{m}} = \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{H}} \cdot \boldsymbol{b}_0 + (\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{A}}) \cdot \mathrm{d}_\nu \boldsymbol{b}_0 - \tfrac{1}{2} \mathrm{d}_\nu (\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{A}}) \cdot \boldsymbol{b}_0 + \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{A}} \cdot \boldsymbol{b}_1 \,. \tag{70}$$

The nonlinear term is given by

$$D_{\mathrm{nl},i} = \tilde{\epsilon}_{ij}^{(2)} (\boldsymbol{a}_0^* \cdot \boldsymbol{a}_0) a_i \,, \tag{71}$$

where $\tilde{\epsilon}^{(2)}$ represent the Fourier transform of a nonlinear dielectric tensor.

Equating terms of order δ^0 in Maxwell's equations, we obtain

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{a}_1 - \mathrm{i}\omega \boldsymbol{b}_1 = -\boldsymbol{\nabla} \times \boldsymbol{b}_0 - \mathrm{d}_t \boldsymbol{b}_0 \,, \tag{72}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{b}_1 + \frac{\mathrm{i}\omega}{c^2}\, \boldsymbol{a}_1 = -\boldsymbol{\nabla} \times \boldsymbol{b}_0 + \frac{1}{c^2}\, \mathrm{d}_t \boldsymbol{a}_0 + \mu_0 \boldsymbol{j}_1 - \mathrm{i}\, \frac{\omega}{c^2}\, (\boldsymbol{a}_0^* \cdot \boldsymbol{a}_0) \tilde{\boldsymbol{\epsilon}}^{(2)} \cdot \boldsymbol{a}_0 \,, \tag{73}$$

where $\boldsymbol{j}_1$ represents the total first order current: $\boldsymbol{j}_1 = \boldsymbol{j}_{1\mathrm{e}} + \boldsymbol{j}_{1\mathrm{m}}$. The electric and magnetic current are given in eq. (69) and in eq. (70) respectively. Using eq. (72) to solve for $\boldsymbol{b}_1$, one obtains

$$\boldsymbol{b}_1 = -\frac{\mathrm{i}}{\omega}\, (\boldsymbol{\nabla} \times \boldsymbol{a}_0 + \mathrm{d}_t \boldsymbol{b}_0 + \mathrm{i}\boldsymbol{k} \times \boldsymbol{a}_1) \,. \tag{74}$$

This expression for $\boldsymbol{b}_1$ is now substituted, along with the expression for $\boldsymbol{b}_0$ in eq. (59), into eq. (73), the following equation is obtained:

$$\frac{\boldsymbol{k}}{\omega} \times \left[\nabla \times \boldsymbol{a}_0 + \mathrm{i}\boldsymbol{k} \times \boldsymbol{a}_1 + \mathrm{d}_t\left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right)\right] + \frac{\omega}{c^2}\,\boldsymbol{a}_1 - \mu_0 \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{A}} \cdot \boldsymbol{a}_1 - \mu_0 \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{A}} \cdot \left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right)$$
$$= -\nabla \times \left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right) + \frac{1}{c^2}\,\mathrm{d}_t \boldsymbol{a}_0 + \mu_0 \Big[(\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{A}}) \cdot \mathrm{d}_\nu \boldsymbol{a}_0 - \tfrac{1}{2}\mathrm{d}_\nu(\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{A}}) \cdot \boldsymbol{a}_0$$
$$+ (\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{A}}) \cdot \mathrm{d}_\nu\left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right) - \tfrac{1}{2}\mathrm{d}_\nu(\partial^\nu \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{A}}) \cdot \mathrm{d}_\nu\left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right) + \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{H}} \cdot \boldsymbol{a}_0 + \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{H}} \cdot \left(\frac{\boldsymbol{k} \times \boldsymbol{a}_0}{\omega}\right)\Big]$$
$$- \mathrm{i}\,\frac{\omega}{c^2}\,(\boldsymbol{a}_0^* \cdot \boldsymbol{a}_0)\tilde{\boldsymbol{\epsilon}}^{(2)} \cdot \boldsymbol{a}_0 \,. \tag{75}$$

In a manner similar to the linear case [18], eq. (75) can be transformed into the following equation:

$$\mathrm{i}\boldsymbol{N}^{\mathrm{H}} \cdot \boldsymbol{a}_1 = (\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \mathrm{d}_\nu \boldsymbol{a}_0 - \mathrm{i}\,\frac{\omega}{c^2}\,(\boldsymbol{a}_0^* \cdot \boldsymbol{a}_0)\tilde{\boldsymbol{\epsilon}}^{(2)} \cdot \boldsymbol{a}_0 + \tfrac{1}{2}\mathrm{d}_\nu(\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \boldsymbol{a}_0 + \mu_0 \tilde{\boldsymbol{\sigma}}^{\mathrm{H}} \cdot \boldsymbol{a}_0 \,, \tag{76}$$

where $\tilde{\boldsymbol{\sigma}}^{\mathrm{H}}$ is the total Hermitian part of the conductivity tensor:

$$\tilde{\boldsymbol{\sigma}}^{\mathrm{H}} = \tilde{\boldsymbol{\sigma}}_{\mathrm{e}}^{\mathrm{H}} + \tilde{\boldsymbol{\sigma}}_{\mathrm{m}}^{\mathrm{H}} \times \left(\frac{\boldsymbol{k}}{\omega}\right),$$

We now project eq. (76) onto $\boldsymbol{L}$, where $\boldsymbol{L}$ is the left null unit vector of eq. (61). We can express $\boldsymbol{a}_0$ as

$$\boldsymbol{a}_0 = g(\boldsymbol{x}, t)\,\boldsymbol{R} \,, \tag{77}$$

where $\boldsymbol{R}$ is the corresponding right null unit vector. We also make use of the following equation that is is obtained from the linear eikonal eq. (61):

$$\boldsymbol{N}^{\mathrm{H}} \cdot \boldsymbol{R} = N\boldsymbol{R} \,, \tag{78}$$

where the proper sets of (ω, $\boldsymbol{k}$) are used. Applying the operator $\boldsymbol{L} \cdot \partial^\nu$ to this equation, one obtains

$$\boldsymbol{L} \cdot (\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \boldsymbol{R} = \partial^\nu N \,, \tag{79}$$

which means that the term $\boldsymbol{L} \cdot (\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \boldsymbol{R}\mathrm{d}_\nu$ is a derivative along the ray. In addition, one can write

$$\boldsymbol{L} \cdot (\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \mathrm{d}_\nu \boldsymbol{R} + \tfrac{1}{2}\boldsymbol{L} \cdot \mathrm{d}_\nu(\partial^\nu \boldsymbol{N}^{\mathrm{H}}) \cdot \boldsymbol{R} = \mathrm{d}_\nu[\boldsymbol{L} \cdot (\partial^\nu \boldsymbol{N}) \cdot \boldsymbol{R}] \,. \tag{80}$$

Eq. (76) will then yield the following transport equation:

$$\mathrm{d}_\tau g - \mathrm{i}\,\frac{\omega}{c^2}\,g^2(\boldsymbol{L} \cdot \tilde{\boldsymbol{\epsilon}}^{(2)} \cdot \boldsymbol{R})g + \tfrac{1}{2}\mathrm{d}\nu(\partial^\nu N)g + g(\boldsymbol{L} \cdot \tilde{\boldsymbol{\sigma}}^{\mathrm{H}} \cdot \boldsymbol{R}) = 0 \,. \tag{81}$$

Eq. (81) is a first order quasilinear differential equation that describes the evolution of the zeroth order amplitude. In the linear case the second term of eq. (81) is zero and the linear solution is

$$g = f(\boldsymbol{x}, t) \,, \tag{82}$$

where the function $f(\boldsymbol{x}, t)$ describes the evolution of the amplitude along the rays due to the inhomogeneity of the medium, and to changes in the ray geometry. It is given by

$$f(\boldsymbol{x}, t) = f_0 \exp\Big(-\int_0^{\tau} \mathrm{d}\tau'\, Q\Big)\,, \tag{83}$$

where Q is expressed explicitly as

$$Q = \tfrac{1}{2}\mathrm{d}_\nu(\partial^\nu N) + \boldsymbol{L}\cdot\tilde{\boldsymbol{\sigma}}^{\mathrm{H}}\cdot\boldsymbol{R}\,. \tag{84}$$

In contrast, when the nonlinearity is present, a solution of eq. (81) is given by

$$g = f(\boldsymbol{x}, t)\exp(\mathrm{i}\beta)\,, \tag{85}$$

where f is given in eq. (83) and β is given by

$$\beta = \int_0^{\tau} \frac{\omega}{c^2}\, g^2(\boldsymbol{L}\cdot\tilde{\boldsymbol{\epsilon}}^{(2)}\cdot\boldsymbol{R})\,\mathrm{d}\tau\,. \tag{86}$$

Therefore, the amplitude consists of a slow varying sinusodal part. In the case that the dielectric tensor has an imaginary part, damping may occur.

It should be noted here that in the case of a wave packet with most of its energy concentrated initially close to some wavevector $\boldsymbol{k}$, with higher order dispersive effects retained, terms of the form $(\mathrm{d}^2\omega/\mathrm{d}k_i\,\mathrm{d}k_j)\,\partial_{x_i}\partial_{x_j} g$ can be obtained. A special case of the transport equation obtained will then be the nonlinear Schrödinger equation.

For the problem in which the nonlinearity is of the form:

$$D_{\mathrm{nl},i} = \int \mathrm{d}^3x' \int_{-\infty}^{t} \mathrm{d}t'\, \epsilon_{ijk}^{(2)} E_j E_k\,, \tag{87}$$

the procedure discussed above is still applicable to a first order approximation. In this case a nonlinear phase θ' is obtained:

$$\theta' = \theta + \int_0^{\tau} \mathrm{d}\tau\,(\omega/c^2) L_i(\tilde{\epsilon}_{ijk}^{(2)} R_j R_k) g \exp(\mathrm{i}\theta')\,. \tag{88}$$

4. Conclusion

We have applied geometrical optics to wave propagation in a weakly inhomogeneous, weakly nonlinear, and dispersive medium. We have shown that for weak dispersion (dispersion that is comparable to the nonlinearity), defined by the ordering $\lambda_{\mathrm{De}}^2 \approx \mathcal{O}(\delta^3)$, the zeroth order rays are linear and nondispersive. The wave form, however, is different than that of the linear case. It obeys a KdV -like transport equation. We provided special solutions to this equation. For strong dispersion, the zeroth order rays are dispersive. The amplitude may has the same form as in the linear case except that it is multiplied by a slow varying sinusoidal part.

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