SIAM J. MATH. ANAL.
Vol. 16, No. 2, March 1985

011

n-SERIES PROBLEMS AND THE COUPLING OF ELECTROMAGNETIC WAVES TO APERTURES: A RIEMANN–HILBERT APPROACH*

RICHARD W. ZIOLKOWSKI†

Abstract. An effective approach to the solution of a large class of mixed boundary value problems (those reducible to an n-series problem) is developed. The method is based on the deduction of the equivalent Riemann–Hilbert problem and its solution. This generalized n-series approach leads to analytical descriptions of the coupling of electromagnetic waves through apertures in canonical structures into open or enclosed regions. In particular, it is applied to the canonical problem of plane wave coupling to an infinite circular cylinder with multiple infinite axial slots. Numerical results for currents induced by an H-polarized plane wave on a circular cylinder with a single slit are given.

1. Introduction. Mixed boundary value problems occur in many areas of physics and engineering. A particular class, the electromagnetic and acoustic coupling problems as they apply to an enclosed region, an external source and a coupling aperture, are of major importance, both theoretically and from a practical point of view. Nonetheless, the separable geometries in which one might expect to obtain an analytic solution have not been amenable to treatment until recently, and purely numerical techniques present difficulties largely due to the edge at the rim. Moreover, approximate solutions, such as the one developed by Bethe [1], are limited in their range of applicability.

Techniques borrowed from the analysis of the Riemann–Hilbert problem of complex variable theory and recent developments [2]–[5] in the theory and applications of dual series equations have made it possible to obtain analytical solutions to families of canonical problems descriptive of electromagnetic and acoustic coupling via apertures into enclosed and open regions. Examples of the canonical problems amenable to solution by these techniques include a plane wave incident (with an arbitrary angle of incidence) on a perfectly conducting diffraction grating, on a perfectly conducting circular cylinder with an infinite axial slot, and on a perfectly conducting spherical shell with a circular aperture. They all involve a scattering body with a single aperture (the unit cell of the grating corresponds to the slitted cylinder). Canonical problems involving structures with $(n-1)$-apertures ($n \geqq 2$) require the solution of n-series problems. For instance, the coupling to a cylinder with two axial slots is described by a triple series equations problem.

These coupling problems constitute only a small subset of a large class of mixed boundary value problems that can be reduced to equivalent n-series problems. Standard techniques available from potential theory, such as the ones described in connection with the dual and triple series equations in [6], are cumbersome and are tailored to specific problems. On the other hand, the Riemann–Hilbert problem techniques provide a unified, systematic approach to these equations. The resultant general n-series approach is applicable to all separable geometries. Therefore, it represents a generalization of the Wiener-Hopf method.

* Received by the editors March 22, 1983.

† Electronics Engineering Department, Lawrence Livermore National Laboratory, Livermore, California 94550. This work was supported under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract W-7405-ENG-48.

It has been brought to the author's attention recently that the Riemann–Hilbert problem techniques have actually been applied in this manner to the dual series equations problems of the diffraction grating [7] and the slitted cylinder [8]. Nonetheless, the parallel approach to the general classes of n-series problems that will be discussed in this paper does not appear to have been reported. Several of the notations in this paper were chosen to resemble those employed in [7] for convenient reference.

The connections between the Riemann–Hilbert problem, n-series problems and the electromagnetic coupling through an aperture will be made in this paper. In particular, in §2 the solution of a general class of n-series problems is developed with Riemann–Hilbert problem techniques. A brief review of the Riemann–Hilbert problem itself is included in the appendix for completeness. The application of the resultant generalized n-series approach to the electromagnetic aperture coupling problem is discussed in §3. Analytic solutions for the coupling of E-polarized and H-polarized plane waves to a perfectly conducting infinite circular cylinder with multiple infinite axial slots are derived. Typical numerical results for the currents induced by an H-polarized plane wave on a circular cylinder with a single axial slit are described. Various comments concerning the main aspects of the generalized n-series approach are given in §4.

2. The Riemann–Hilbert approach to n-series problems. As shown in [6], there are many generic problems of the dual and triple series equations type. Only the n-series canonical problems encompassing those related to the slitted cylinder examples to be discussed below will be considered. They are sufficient to illustrate the proposed Riemann–Hilbert approach. The solutions to other generic classes of problems can be inferred from these results.

The Riemann–Hilbert problem, as described in the appendix, is concerned with finding the analytic function that satisfies a prescribed transition condition across an open or a closed curve. Let the unit circle S^1 be divided into two sets, Γ and L, the closure of Γ being the complement of L in S^1, and let each of these sets consist of $(n-1)$, $n \geqq 2$, open nonintersecting segments: $\Gamma=\{\Gamma_1,\cdots,\Gamma_{n-1}\}$ and $L=\{L_1,\cdots,L_{n-1}\}$. Also let $I(\Gamma)=\{I(\Gamma_1),\cdots,I(\Gamma_{n-1})\}$ and $I(L)=\{I(L_1),\cdots,I(L_{n-1})\}$ be the angular decomposition of the interval $[0,2\pi]$ corresponding to those sets. In particular, set

$$\Gamma_j=\left\{e^{i\phi}|\phi\in I(\Gamma_j)=(\theta_{2j-2},\theta_{2j-1})\right\} \qquad (j=1,\cdots,n-1), \tag{2.1a}$$

$$L_j=\left\{e^{i\phi}|\phi\in I(L_j)=(\theta_{2j-1},\theta_{2j})\right\} \qquad (j=1,\cdots,n-1). \tag{2.1b}$$

Consider first the basic n-series problem ($n \geqq 2$):

$$\sum_{m=-\infty}^{\infty} a_m e^{im\phi}=0, \qquad \phi\in I(L), \tag{2.2a}$$

$$\sum_{m=-\infty}^{\infty} \varepsilon_m |m| a_m e^{im\phi}=\xi a_0+f(\phi), \qquad \phi\in I(\Gamma). \tag{2.2b}$$

Depending on the specific problem, $\varepsilon_m=\operatorname{sgn}(m)$ or $\varepsilon_m=[\operatorname{sgn}(m)]^2\equiv+1$, where it is assumed that

$$\operatorname{sgn}(m)=\begin{cases} +1 & \text{for } m\geqq 0,\\ -1 & \text{for } m<0.\end{cases} \tag{2.3}$$

It can be reduced to a Riemann–Hilbert problem as follows. Differentiating (2.2) with respect to ϕ and substituting $x_m = ma_m (m \neq 0)$ in both (2.2a) and (2.2b), one obtains the modified n-series problem

$$\begin{cases} \sum_{m\neq 0} x_m e^{im\phi} = 0, & \phi \in I(L), \quad (2.4a) \\ \sum_{m\neq 0} \varepsilon_m x_m \frac{|m|}{m} e^{im\phi} = \xi a_0 + f(\phi), & \phi \in I(\Gamma). \quad (2.4b) \end{cases}$$

The symbol $\Sigma_{m\neq 0}$ indicates the sum over all terms from $m = -\infty$ to $m = +\infty$ except the term with $m = 0$. Now, introduce the functions

$$x_+(z) = \sum_{m>0} x_m z^m, \tag{2.5a}$$

$$x_-(z) = -\sum_{m<0} x_m z^m \tag{2.5b}$$

which are assumed to be analytic, respectively, on the interior and the exterior of the unit circle S^1. The n-series equations (2.4) can then be rewritten as

$$\begin{cases} x_+(\lambda) - x_-(\lambda) = 0, & \lambda \in L, \quad (2.6a) \\ x_+(\gamma) - T(\gamma) x_-(\gamma) = F(\gamma), & \gamma \in \Gamma, \quad (2.6b) \end{cases}$$

where

$$T(e^{i\phi}) = \begin{cases} +1 & \text{for } \varepsilon_m = \operatorname{sgn}(m), \\ -1 & \text{for } \varepsilon_m = +1, \end{cases} \tag{2.7}$$

and

$$F(e^{i\phi}) = \xi a_0 + f(\phi), \qquad \phi \in I(\Gamma). \tag{2.8}$$

Equation (2.6a) means that $x_+(z)$ and $x_-(z)$ coincide on L, i.e., they continue analytically across L and thus become the same analytic function,

$$x(z) = \begin{cases} x_+(z), & |z| < 1, \\ x_-(z), & |z| > 1. \end{cases} \tag{2.9}$$

Similarly, the functions $x_+(\gamma)$ and $x_-(\gamma)$ in (2.6b) represent, respectively, the limiting values on Γ from the interior and the exterior of S^1 of the same analytic function (2.9); hence (2.6b) describes a discontinuity in that function across the open curve Γ.

It is assumed that the solution $x(z)$ has singularities of order $+\frac{1}{2}$ at each of the endpoints $\alpha_j = \exp(i\theta_{2j-2})$, $\beta_j = \exp(i\theta_{2j-1})$ of $\Gamma_j (j = 1, \cdots, n-1)$ and is zero at infinity. This properly models the behavior of the solution in the electromagnetics case near the edges of the aperture and at infinity. Moreover, for the moment, let the transition function F be a least Hölder continuous on S^1. As indicated in [9], the Riemann–Hilbert problem techniques can actually handle solutions with other singularities, e.g., any of those whose order lies in the interval (0, 1), with a nonzero behavior at infinity and with a transition function satisfying a relaxed continuity condition.

Rewriting (2.6b) as the transition condition

$$x_+(\gamma) = T(\gamma) x_-(\gamma) + F(\gamma), \qquad \gamma \in \Gamma, \tag{2.10}$$

an inhomogeneous Riemann–Hilbert problem with discontinuous coefficients on an open curve is realized. The factors T and F are called, respectively, the coefficient and the free term of this Riemann–Hilbert problem. Its solution, $x(z)$, is developed in [9, Chapter VI, §42]. This problem is first reduced to one with discontinuous coefficients on the closed curve S^1 by setting

$$T_0(\zeta)=\begin{cases} T(\zeta) & \text{for } \zeta\in\Gamma, \\ +1 & \text{for } \zeta\in L \end{cases} \tag{2.11}$$

and

$$F_0(\zeta)=\begin{cases} F(\zeta) & \text{for } \zeta\in\Gamma, \\ 0 & \text{for } \zeta\in L, \end{cases} \tag{2.12}$$

so that (2.10) becomes

$$x_+(\zeta)=T_0(\zeta)x_-(\zeta)+F_0(\zeta), \qquad \zeta\in S^1. \tag{2.13}$$

Next the problem is reduced to one with continuous coefficients by introducing the characteristic function $[1/G(z)]$ of the problem, i.e., the function that has the same singular behavior as $x(z)$ at the endpoints (α_j,β_j) of the segments $\Gamma_j(j=1,\cdots,n-1)$, and which makes the product xG nonsingular at those points, and satisfies the homogeneous Riemann–Hilbert problem

$$1/G_+(\zeta)=T_0(\zeta)/G_-(\zeta), \qquad \zeta\in S^1, \tag{2.14a}$$

Note that (2.14a) also means

$$T_0(\zeta)=G_-(\zeta)/G_+(\zeta), \qquad \zeta\in S^1. \tag{2.14b}$$

Thus, mulitplying (2.13) by $G_+(\zeta)$ and defining the functions

$$\Phi(z)=x(z)G(z), \tag{2.15}$$

$$\Psi(z)=G_+(z)F_0(z), \tag{2.16}$$

one obtains

$$\Phi_+(\zeta)=\Phi_-(\zeta)+\Psi(\zeta), \qquad \zeta\in S^1. \tag{2.17}$$

This represents the transition condition of a Riemann–Hilbert problem with continuous coefficients on a closed curve. Its solution is simply [9, pp. 96–99]

$$\Phi(z)=\frac{1}{2\pi i}\int_{S^1}\frac{\Psi(\tau)\,d\tau}{\tau-z}+P_{n-2}(z), \tag{2.18}$$

where $P_{n-2}(z)$ is a polynomial of degree $(n-2)$ in z:

$$P_{n-2}(z)=c_0+c_1z^1+\cdots+c_{n-2}z^{n-2}. \tag{2.19}$$

Consequently, the desired solution of (2.13) is defined as

$$x(z)=\frac{1}{2\pi i}\frac{1}{G(z)}\int_\Gamma\frac{G_+(\tau)F(\tau)\,d\tau}{\tau-z}+\frac{1}{G(z)}P_{n-2}(z). \tag{2.20}$$

The procedure to obtain the characteristic function or equivalently the function G is given in [9, §42]. It depends on the index of the coefficient $T_0(\zeta)$, the index of the problem. It is readily shown that the index is $(n-1)$ for the present problem and that

$$(2.21)\qquad G(z)=\begin{cases}\prod_{j=1}^{n-1}\left|(z-\alpha_j)(z-\beta_j)\right|^{1/2} & \text{for } \varepsilon_m=\operatorname{sgn}(m),\\ \prod_{j=1}^{n-1}\left[(z-\alpha_j)(z-\beta_j)\right]^{1/2} & \text{for } \varepsilon_m+1.\end{cases}$$

The results for the $\varepsilon_m=+1$ case are presented explicitly in [9, §42.2]. For that case the branches of G will be chosen so that as $z\to\gamma\in\Gamma$ from the interior of S^1: $G(z)\to G_+(\gamma)$, and from its exterior: $G(z)\to G_-(\gamma)=-G_+(\gamma)$. This choice satisfies the restriction of (2.14b) to Γ:

$$(2.22)\qquad G_-(\gamma)=T(\gamma)G_+(\gamma),\qquad \gamma\in\Gamma.$$

The polynomial term, $P_{n-2}(z)$, in (2.18) and (2.20) is introduced to account for the assumed behavior of x at infinity. In particular, as $|z|\to\infty$ (2.21) yields

$$(2.23)\qquad |G(z)|\sim|z|^{n-1}.$$

Therefore, in that limit the magnitude of the solution

$$(2.24)\qquad |x(z)|\sim\frac{|P_{n-2}(z)|}{|G(z)|}\sim\frac{|c_{n-2}|}{|z|}\to 0$$

as desired.

The solution (2.20) provides a means to generate another relation between the limiting values x_+ and x_- on Γ. Let

$$(2.25)\qquad \Omega(z)=\frac{1}{2\pi i}P\int_\Gamma\frac{G_+(\tau)F(\tau)\,d\tau}{\tau-z},$$

where $P\int$ means to take the Cauchy principal value of the integral. The Plemelj–Sokhotskii formulas [see (A.3) in the appendix] together with (2.20) and (2.22) give

$$(2.26)\qquad x_+(\gamma)+T(\gamma)x_-(\gamma)=2[\Omega(\gamma)+P_{n-2}(\gamma)]/G_+(\gamma),\qquad \gamma\in\Gamma.$$

The coefficients $x_m (m\neq 0)$ and the constants $a_0, c_0,\cdots,c_{n-2}$ can now be obtained as follows.

First consider the case in which $\varepsilon_m=\operatorname{sgn}(m)$. Combining (2.5), (2.6) and (2.12), one obtains for all $\zeta=e^{i\phi}\in S^1$:

$$(2.27)\qquad x_+(e^{i\phi})-x_-(e^{i\phi})=\sum_{m\neq 0}x_m e^{im\phi}=F_0(e^{i\phi}).$$

Fourier inversion of this expression gives the terms

$$(2.28a)\qquad x_m=\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{-im\phi}F_0(e^{i\phi})=\frac{1}{2\pi}\int_\Gamma d\phi\, e^{-im\phi}F(e^{i\phi})\qquad (m\neq 0),$$

$$(2.28b)\qquad a_0=\frac{-\int_\Gamma f(\phi)\,d\phi}{\xi\int_\Gamma d\phi}.$$

If the solution (2.20) is desired, the constants $c_0, \cdots, c_{n-2}$ are then obtained from a system of $(n-1)$ equations:

$$\sum_{m\neq 0} \frac{|m|}{m} x_m e^{im\gamma_j} = 2\left[\Omega(e^{i\gamma_j}) + P_{n-2}(e^{i\gamma_j})/G_+(e^{i\gamma_j})\right] \qquad (j=1,\cdots,n-1), \tag{2.29a}$$

derived by evaluating the relation (2.26) at the midpoints

$$\gamma_j = \frac{1}{2}(\theta_{2j-1} + \theta_{2j-2}) \tag{2.29b}$$

of the intervals $I(\Gamma_j)(j=1,\cdots,n-1)$.

On the other hand, for the case in which $\varepsilon_m = +1$ the combination of (2.5), (2.6a) and (2.26) yields for all $\zeta = e^{i\phi} \in S^1$:

$$x_+(e^{i\phi}) - x_-(e^{i\phi}) = \sum_{m\neq 0} x_m e^{im\phi} = 2g(e^{i\phi})\left[\Omega(e^{i\phi}) + P_{n-2}(e^{i\phi})\right], \tag{2.30}$$

where

$$g(\zeta) = \begin{cases} 1/G_+(\zeta) & \text{for } \zeta \in \Gamma, \\ 0 & \text{for } \zeta \in L. \end{cases} \tag{2.31}$$

Defining the terms

$$v(\zeta) = \frac{1}{i\pi} P\int_\Gamma \frac{G_+(\tau)\,d\tau}{\tau - \zeta}, \tag{2.32a}$$

$$V(\zeta) = \frac{1}{i\pi} P\int_\Gamma \frac{G_+(\tau) f(\tau)\,d\tau}{\tau - \zeta}, \tag{2.32b}$$

$$v_m = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\phi}) g(e^{i\phi}) e^{-im\phi}\,d\phi = \frac{1}{2\pi}\int_\Gamma \frac{v(e^{i\phi}) e^{-im\phi}\,d\phi}{G_+(e^{i\phi})}, \tag{2.32c}$$

$$V_m = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\phi}) g(e^{i\phi}) e^{-im\phi}\,d\phi = \frac{1}{2\pi}\int_\Gamma \frac{V(e^{i\phi}) e^{-im\phi}\,d\phi}{G_+(e^{i\phi})}, \tag{2.32d}$$

$$R_m = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\phi}) e^{-im\phi}\,d\phi = \frac{1}{2\pi}\int_\Gamma \frac{e^{-im\phi}\,d\phi}{G_+(e^{i\phi})}, \tag{2.32e}$$

Fourier inversion of (2.30) yields a linear system of equations for the coefficients a_0 and $x_m (m\neq 0)$ of the form:

$$x_m = \xi a_0 v_m + V_m + 2\sum_{j=0}^{n-2} c_j R_{m-j} \qquad (m\neq 0), \tag{2.33a}$$

$$0 = \xi a_0 v_0 + V_0 + 2\sum_{j=0}^{n-2} c_j R_{-j} \qquad (m=0). \tag{2.33b}$$

This system is completed by the $(n-1)$ relations

$$a_0 = -\sum_{m\neq 0} \frac{x_m}{m} e^{im\psi_l} \qquad (l=1,\cdots,n-1) \tag{2.34a}$$

obtained from (2.2a) by setting ϕ equal to ψ_l, the midpoint of the interval $I(L_l)$:

(2.34b) $$\psi_l=\frac{1}{2}(\theta_{2l-1}+\theta_{2l}).$$

With (2.33a) this constraint system becomes

(2.35) $$-a_0=\xi a_0 w_l+W_l+2\sum_{j=0}^{n-2} c_j S_l^j \qquad (l=1,\cdots,n-1),$$

where

(2.36a) $$w_l=\sum_{m\neq 0}\frac{v_m}{m}e^{im\psi_l},$$

(2.36b) $$W_l=\sum_{m\neq 0}\frac{V_m}{m}e^{im\psi_l},$$

(2.36c) $$S_l^j=\sum_{m\neq 0}\frac{R_{m-j}}{m}e^{im\psi_l}.$$

Note that the introduction of the $(n-1)$ constraint relations (2.29) and (2.34) is necessitated by the appearance of the $n-1$ constants $c_0,\cdots,c_{n-2}$ in the Riemann–Hilbert solution (2.20). They have, however, a direct effect only on the solution of the n-series problem (2.2) with $\varepsilon_m=+1$. Furthermore, the choice of those particular relations is somewhat arbitrary. Their evaluation at any one point in each of the intervals $I(\Gamma_j)$ and $I(L_j)(j=1,\cdots,n-1)$ instead of the angles γ_j and $\psi_j(j=1,\cdots,n-1)$ would equally suffice. Nonetheless, the midpoint rule is systematic and computationally convenient.

These general results are considerably simplified if the forcing function f has the Fourier expansion:

(2.37) $$f(\phi)=\sum_{n=-\infty}^{\infty} f_n e^{in\phi}.$$

Defining the additional coefficient

(2.38) $$Q_m=\frac{1}{2\pi}\int_\Gamma d\phi e^{-im\phi},$$

the solution system (2.28) becomes

(2.39a) $$ma_m=\xi a_0 Q_0+\sum_{n=-\infty}^{\infty} f_n Q_{m-n},$$

(2.39b) $$a_0=-\sum_{n=-\infty}^{\infty} f_n Q_{-n}/\xi Q_0.$$

Similarly, defining the coefficients

(2.40a) $$V_n(e^{i\phi})=\frac{1}{i\pi}P\int_\Gamma\frac{G_+(\tau)\tau^n d\tau}{\tau-e^{i\phi}},$$

(2.40b) $$V_m^n=\frac{1}{2\pi}\int_0^{2\pi}V_n(e^{i\phi})g(e^{i\phi})e^{-im\phi}d\phi=\frac{1}{2\pi}\int_\Gamma\frac{V_n(e^{i\phi})e^{-im\phi}d\phi}{G_+(e^{i\phi})},$$

$$W_l^n=\sum_{m\neq 0} V_m^n\frac{e^{im\psi_l}}{m}, \tag{2.40c}$$

the solution system (2.33) and (2.35) becomes

$$ma_m=\xi a_0V_m^0+\sum_{n=-\infty}^{\infty} f_nV_m^n+2\sum_{l=0}^{n-2} c_lR_{m-l} \qquad (m\neq 0), \tag{2.41a}$$

$$0=\xi a_0V_0^0+\sum_{n=-\infty}^{\infty} f_nV_0^n+2\sum_{l=0}^{n-2} c_lR_{-l} \qquad (m=0), \tag{2.41b}$$

$$0=(1+\xi W_l^0)a_0+\sum_{n=-\infty}^{\infty} f_nW_l^n+2\sum_{j=0}^{n-2} c_jS_l^j \qquad (l=1,\cdots,n-1). \tag{2.41c}$$

Note that (2.41b) and (2.41c) can be solved simultaneously to obtain the n constants a_0, $c_0,\cdots,c_{n-2}$ and then the values $a_m(m\neq 0)$ follow immediately from (2.41a) or (2.41a), (2.41b) and (2.41c) can be solved simultaneously as an entire system. However, because (2.41b) and (2.41c) are decoupled from (2.41a), the former would be numerically superior to the latter.

The solution to the basic n-series problem

$$\sum_{m=-\infty}^{\infty} b_me^{im\phi}=h(\phi), \qquad \phi\in I(L), \tag{2.42a}$$

$$\sum_{m=-\infty}^{\infty} \varepsilon_mb_m|m|e^{im\phi}=0, \quad \phi\in I(\Gamma), \tag{2.42b}$$

which is complementary to the one defined by (2.2), follows in an analogous manner. Let

$$y_+(z)=\sum_{m=0}^{\infty} b_mz^m, \tag{2.43a}$$

$$y_-(z)=\sum_{m<0} \varepsilon_mb_mz^m. \tag{2.43b}$$

Integrating (2.42b) and setting

$$b_0=-\sum_{m\neq 0}\varepsilon_m\frac{|m|}{m}b_me^{im\gamma_l} \qquad (l=1,\cdots,n-1). \tag{2.44}$$

Equations (2.42) can then be rewritten as

$$y_+(\lambda)-T(\lambda)y_-(\lambda)=h(\phi), \qquad \lambda=e^{i\phi}\in L, \tag{2.45a}$$

$$y_+(\gamma)-y_-(\gamma)=0, \qquad \gamma\in\Gamma. \tag{2.45b}$$

The $(n-1)$ relations (2.44) are analogous to the constraint system (2.34). Moreover, the system (2.45) has the same form as (2.6) except that the line of discontinuity is now L rather than Γ. Consequently, the characteristic function $\tilde{G}(z)$ of the corresponding Riemann-Hilbert problem is (2.21) with α_j and β_j, the endpoints of the arcs Γ_j, replaced with $\tilde{\alpha}_j=\exp(i\theta_{2j-1})$ and $\tilde{\beta}_j=\exp(i\theta_{2j})$, the endpoints of the complementary arcs L_j. However, since S^1 is a closed curve, $\theta_{2(n-1)} \equiv \theta_0$. Therefore, $\tilde{G} \equiv G$. Assuming the forcing function h has the Fourier expansion

$$h(\phi) = \sum_{n=-\infty}^{\infty} h_n e^{in\phi}, \tag{2.46}$$

the preceding Riemann–Hilbert techniques then yield for the $\varepsilon_m = \operatorname{sgn}(m)$ case the solution coefficients

$$b_m = \frac{1}{2\pi} \int_L h(\phi) e^{-im\phi} d\phi = \sum_{n=-\infty}^{\infty} h_n \tilde{Q}_{m-n} \qquad (\text{all } m), \tag{2.47a}$$

where

$$\tilde{Q}_m = \frac{1}{2\pi} \int_L e^{-im\phi} d\phi, \tag{2.47b}$$

and for the $\varepsilon_m = +1$ case the solution system

$$\operatorname{sgn}(m) b_m = \sum_{n=-\infty}^{\infty} h_n \tilde{V}_m^n + 2 \sum_{j=0}^{n-2} c_j \tilde{R}_{m-j} \qquad (\text{all } m), \tag{2.48a}$$

$$-b_0 = \sum_{n=-\infty}^{\infty} h_n \tilde{W}_l^n + 2 \sum_{j=0}^{n-2} c_j \tilde{S}_l^j \qquad (l = 1, \cdots, n-1), \tag{2.48b}$$

where

$$\tilde{V}_n(e^{i\phi}) = \frac{1}{i\pi} P \int_L \frac{G_+(\tau)\tau^n d\tau}{\tau - e^{i\phi}}, \tag{2.49a}$$

$$\tilde{V}_m^n = \frac{1}{2\pi} \int_L \frac{\tilde{V}_n(e^{i\phi}) e^{-im\phi} d\phi}{G_+(e^{i\phi})}, \tag{2.49b}$$

$$\tilde{R}_m = \frac{1}{2\pi} \int_L \frac{e^{-im\phi} d\phi}{G_+(e^{i\phi})}, \tag{2.49c}$$

$$\tilde{W}_l^n = \sum_{m \neq 0} \tilde{V}_m^n e^{im\gamma_l}, \tag{2.49d}$$

$$\tilde{S}_l^j = \sum_{m \neq 0} \tilde{R}_{m-j} e^{im\gamma_l}. \tag{2.49e}$$

Note that the solution of the general n-series problem

$$\sum_{m=-\infty}^{\infty} c_m e^{im\phi} = h(\phi), \qquad \phi \in I(L), \tag{2.50a}$$

$$\sum_{m=-\infty}^{\infty} \varepsilon_m c_m |m| e^{im\phi} = \xi a_0 + f(\phi), \qquad \phi \in I(\Gamma) \tag{2.50b}$$

can now be obtained. Solving independently the problems defined by (2.2) and (2.42), the solution of (2.50) follows immediately by setting $c_m = a_m + b_m$.

More complicated n-series systems of the form

$$\sum_{m=-\infty}^{\infty} a_m e^{im\phi}=0, \qquad \phi\in I(L), \tag{2.51a}$$

$$\sum_{m=-\infty}^{\infty} \varepsilon_m a_m |m| \tau_m e^{im\phi}=\xi a_0+f(\phi), \qquad \phi\in I(\Gamma) \tag{2.51b}$$

and

$$\sum_{m=-\infty}^{\infty} b_m e^{im\phi}=h(\phi), \qquad \phi\in I(L), \tag{2.52a}$$

$$\sum_{m=-\infty}^{\infty} \varepsilon_m b_m |m| \tau_m e^{im\phi}=0, \qquad \phi\in I(\Gamma) \tag{2.52b}$$

are encountered in mixed boundary value problems such as those describing aperture coupling. Assuming that the coefficient function τ_m has the decomposition

$$\tau_m=1+\chi_m, \tag{2.53}$$

where the function χ_m satisfies the limiting condition

$$\lim_{|m|\to\infty} \chi_m \sim o\left(\frac{1}{|m|}\right), \tag{2.54}$$

these n-series problems can be reduced to the basic problems (2.2) and (2.42) by treating the χ_m dependent terms as forcing functions. In particular, define the functions $\tilde{y}_+(z)$ and $\tilde{y}_-(z)$ by (2.43a) and (2.43b) respectively, with b_m replaced by the modified coefficient

$$\tilde{b}_m=b_m\tau_m, \tag{2.55}$$

and define the modified forcing functions

$$\tilde{F}(e^{i\phi})=\xi a_0+\sum_{n=-\infty}^{\infty} \tilde{F}_n e^{in\phi} \tag{2.56a}$$

and

$$\tilde{H}(e^{i\phi})=\sum_{n=-\infty}^{\infty} \tilde{H}_n e^{in\phi}, \tag{2.57a}$$

where the Fourier coefficients

$$\tilde{F}_n=f_n-|n|a_n\chi_n\equiv f_n-\frac{|n|}{n}x_n\chi_n \tag{2.56b}$$

and

$$\tilde{H}_n=h_n+b_n\chi_n=h_n+\frac{\tilde{b}_n\chi_n}{1+\chi_n}. \tag{2.57b}$$

Systems (2.51) and (2.52) can then be replaced by the equations

$$x_+(\lambda)-x_-(\lambda)=0, \qquad \lambda\in L, \tag{2.58a}$$
$$x_+(\gamma)-T(\gamma)x_-(\gamma)=\tilde{F}(\gamma), \qquad \gamma\in\Gamma, \tag{2.58b}$$

and

$$\tilde{y}_+(\lambda)-T(\lambda)\tilde{y}_-(\lambda)=\tilde{H}(\lambda), \qquad \lambda\in L, \tag{2.59a}$$
$$\tilde{y}_+(\gamma)-\tilde{y}_-(\gamma)=0, \qquad \gamma\in\Gamma. \tag{2.59b}$$

The associated constraint relations in the $\varepsilon_m=+1$ case are (2.34) and (2.44), the latter having each b_m replaced with $\tilde{b}_m$. The Riemann–Hilbert technique can now be applied to (2.58) and (2.59). The solutions for the $\varepsilon_m=\operatorname{sgn}(m)$ cases:

$$\begin{aligned} x_m&=\xi a_0Q_0+\sum_{n=-\infty}^{\infty} f_nQ_{m-n}-\sum_{n=-\infty}^{\infty}\frac{|n|}{n}x_n\chi_nQ_{m-n} && (m\neq 0),\\ a_0&=\left[\sum_{n=-\infty}^{\infty}\frac{|n|}{n}x_n\chi_nQ_{-n}-\sum_{n=-\infty}^{\infty} f_nQ_{-n}\right]\Big/\ \xi Q_0 && (m=0), \end{aligned} \tag{2.60}$$

$$\tilde{b}_m=\sum_{n=-\infty}^{\infty} h_n\tilde{Q}_{m-n}+\sum_{n=-\infty}^{\infty}\frac{\tilde{b}_n\chi_n\tilde{Q}_{m-n}}{1+\chi_n} \qquad (\text{all } m), \tag{2.61}$$

and for the $\varepsilon_m=+1$ cases:

(2.62)

$$x_m=\xi a_0V_m^0+\sum_{n=-\infty}^{\infty} f_nV_m^n-\sum_{n=-\infty}^{\infty}\frac{|n|}{n}x_n\chi_nV_m^n+2\sum_{j=0}^{n-2}c_jR_{m-j} \qquad (m\neq 0),$$

$$0=\xi a_0V_0^0+\sum_{n=-\infty}^{\infty} f_nV_0^n-\sum_{n=-\infty}^{\infty}\frac{|n|}{n}x_n\chi_nV_0^n+2\sum_{j=0}^{n-2}c_jR_{-j} \qquad (m=0),$$

$$0=(1+\xi W_l^0)a_0+\sum_{n=-\infty}^{\infty} f_nW_l^n-\sum_{n=-\infty}^{\infty}\frac{|n|}{n}x_n\chi_nW_l^n+2\sum_{j=0}^{n-2}c_jS_l^j \qquad (l=1,\cdots,n-1),$$

$$\begin{aligned} \operatorname{sgn}(m)\tilde{b}_m&=\sum_{n=-\infty}^{\infty} h_n\tilde{V}_m^n+\sum_{n=-\infty}^{\infty}\frac{\tilde{b}_n\chi_n\tilde{V}_m^n}{1+\chi_n}+2\sum_{j=0}^{n-2}c_j\tilde{R}_{-j} && (\text{all } m),\\ -\tilde{b}_0&=\sum_{n=-\infty}^{\infty} h_n\tilde{W}_l^n+\sum_{n=-\infty}^{\infty}\frac{\tilde{b}_n\chi_n\tilde{W}_l^n}{1+\chi_n}+2\sum_{j=0}^{n-2}c_j\tilde{S}_l^j && (l=1,\cdots,n-1) \end{aligned} \tag{2.63}$$

follow immediately from (2.39), (2.41), (2.47) and (2.48).

Note that all of these solution systems have the general form

$$u_m=\sum_{n=-\infty}^{\infty}\Lambda_{mn}u_n+v_m, \qquad m=0,\pm 1,\pm 2,\cdots \tag{2.64}$$

where the matrix Λ_{mn} and the vector v_m are known quantities. This infinite system of equations represents a Fredholm equation of the second kind and may be treated with a variety of methods. The technique utilized in the slitted cylinder examples will be described in the next section. It also should be noted that the assumption (2.54) is made because it causes, for example, the combination $x_n \chi_n$ to behave as $a_n n^{-\varepsilon}$, $\varepsilon > 0$, as $n \to \infty$, thereby insuring the convergence of the associated sums in (2.60) and (2.62). If desired, this condition could be relaxed. It suffices for the slitted cylinder examples.

Finally, as a further generalization of the above results, consider, for instance, the n-series problem

$$(2.65) \qquad \begin{cases} \sum_{j=-\infty}^{\infty} a_j e_j(\phi) = 0, & \phi \in I(L), \\ \sum_{j=-\infty}^{\infty} a_j \tilde{e}_j(\phi) = f(\phi), & \phi \in I(\Gamma), \end{cases}$$

where the functions $\{e_m(\phi), m = 0, 1, \cdots\}$ form an orthonormal basis of the Hilbert space $\mathscr{L}_2([0, 2\pi])$. Since this system is constructed from the mixed boundary conditions, the functions $\{\tilde{e}_m(\phi)\}$ must be linear combinations of the basis functions $\{e_m(\phi)\}$. Moreover, because the set $\{e^{im\phi}, m = 0, \pm 1, \cdots\}$ is also a basis of $\mathscr{L}_2([0, 2\pi])$, each function $e_m(\phi)$; hence, each $\tilde{e}_m(\phi)$ can be expanded in terms of those basis functions. In particular, set

$$(2.66a) \qquad e_j(\phi) = \sum_{m=-\infty}^{\infty} U_{jm} e^{im\phi},$$

$$(2.66b) \qquad \tilde{e}_j(\phi) = \sum_{m=-\infty}^{\infty} \eta(m) U_{jm} e^{im\phi}.$$

Thus, defining

$$(2.67) \qquad x_m = \sum_{j=-\infty}^{\infty} U_{jm} a_j,$$

the n-series system (2.65) becomes, for example,

$$(2.68a) \qquad \sum_{m=-\infty}^{\infty} x_m e^{im\phi} = 0, \qquad \phi \in I(L),$$

$$(2.68b) \qquad \sum_{m=-\infty}^{\infty} x_m \operatorname{sgn}(m) e^{im\phi} = F(\phi), \qquad \phi \in I(\Gamma),$$

where the Fourier coefficients of the forcing function F are

$$(2.69) \qquad F_n = f_n + [\operatorname{sgn}(n) - \eta(n)] x_n.$$

The solution to the system (2.68) follows immediately from the preceding results.

3. Electromagnetic coupling to a slitted cylinder. A variety of problems including those describing the coupling of electromagnetic waves to an enclosed region can be reduced to an n-series problem. For instance, if the shape of the scattering body coincides with a constant coordinate surface in one of the coordinate systems for which the vector field equations are separable, the incident and scattered fields are first expanded in terms of the corresponding eigenfunctions. The n-series equations are then realized by enforcing on that surface the boundary conditions for the tangential electric and magnetic fields over the *aperture* and on the *perfect conductor*.

In particular, consider the electromagnetic coupling of a plane wave to a thin infinite perfectly conducting circular cylinder with $(n-1)$ infinite axial slots. The magnetic field vector of the plane wave is taken to be parallel to the axis of the cylinder. This H-polarized plane wave is assumed normally incident on the cylinder; hence, the problem is two-dimensional. A cylindrical coordinate system (ρ,ϕ,z) is centered on the axis of the cylinder; the z-axis coincides with the cylinder's axis. The angle of incidence, ϕ^{inc}, of the plane wave is arbitrary. The radius of the cylinder is $\mathbf{a}$. The angular extent of the metallic portions of the cylinder coincides with the interval $I(\Gamma)$, the apertures with $I(L)$. This geometry is illustrated in Fig. 1 for a cylinder with a single axial slot $(n=2)$. The currents induced by the plane wave on the metallic portions of the cylinder are desired. This problem is reduced to an n-series problem as follows.

For the given polarization Maxwell's equations decouple and only the E_ρ, E_ϕ and H_z components of the field are excited. The components of the field tangential to the surface of the aperture and the cylinder are of particular importance. They are related by

$$E_\phi=\frac{j}{\omega\varepsilon}\partial_\rho H_z \tag{3.1}$$

where, as throughout this paper, a $e^{j\omega t}$ time dependence is assumed.

The incident magnetic field has the Fourier mode expansion:

$$H_z^{\mathrm{inc}}=A_0e^{jk\rho\cos(\phi-\phi^{\mathrm{inc}})}=A_0\sum_{n=-\infty}^{\infty}\left[j^{|n|}J_{|n|}(k\rho)e^{-jn\phi^{\mathrm{inc}}}\right]e^{jn\phi}. \tag{3.2}$$

From (3.1) it follows that

$$E_\phi^{\mathrm{inc}}=jZ_0A_0\sum_{n=-\infty}^{\infty}\left[j^{|n|}J'_{|n|}(k\rho)e^{-jn\phi^{\mathrm{inc}}}\right]e^{jn\phi} \tag{3.3}$$

where $J'_m(x)=dJ_m/dx$ and $Z_0=k/\omega\varepsilon$ is the free-space characteristic impedance. The corresponding Fourier expansions of the scattered fields are:

$$H^s_{z>}=A_0\sum_{n=-\infty}^{\infty}a_nJ'_{|n|}(k\mathbf{a})H_{|n|}(k\rho)e^{jn\phi}\qquad(\rho>\mathbf{a}), \tag{3.4a}$$

$$H^s_{z<}=A_0\sum_{n=-\infty}^{\infty}a_nJ_{|n|}(k\rho)H'_{|n|}(k\mathbf{a})e^{jn\phi}\qquad(\rho<\mathbf{a}), \tag{3.4b}$$

$$E^s_{\phi>}=jZ_0A_0\sum_{n=-\infty}^{\infty}a_nJ'_{|n|}(k\mathbf{a})H'_{|n|}(k\rho)e^{jn\phi}\qquad(\rho>\mathbf{a}), \tag{3.4c}$$

$$E^s_{\phi<}=jZ_0A_0\sum_{n=-\infty}^{\infty}a_nJ'_{|n|}(k\rho)H'_{|n|}(k\mathbf{a})e^{jn\phi}\qquad(\rho<\mathbf{a}), \tag{3.4d}$$

where H_n is the Hankel function of second kind and order n and $H'_n(x)=dH_n/dx$. The boundary conditions for the tangential electric and magnetic fields at the surface $\rho=\mathbf{a}$ are now enforced to obtain the n-series equations.

Since the total tangential electric field is zero on the metal, the scattered and the negative of the incident electric fields are equal there:

$$E^s_\phi(\mathbf{a},\phi)=-E^{\mathrm{inc}}_\phi(\mathbf{a},\phi)\equiv A_0\mathbb{E}(\phi)\qquad\phi\in I(\Gamma). \tag{3.5}$$

Substituting (3.4c) or (3.4d) ((3.4c) and (3.4d) guarantee the continuity of the tangential component of the scattered electric field across the interface $\rho=\mathbf{a}$) into (3.5), one obtains

$$jZ_0 \sum_{n=-\infty}^{\infty} a_n J'_{|n|}(k\mathbf{a}) H'_{|n|}(k\mathbf{a}) e^{jn\phi} = \mathbb{E}(\phi), \qquad \phi \in I(\Gamma). \tag{3.6}$$

The *dc* components of the fields can be extracted from this relation by introducing the functions $K_n(x)$ so that

$$-j\pi J'_0(x) H'_0(x) = 1 + K_0(x) \qquad (n=0), \tag{3.7a}$$

$$j\pi x^2 J'_n(x) H'_n(x) = n[1 + K_n(x)] \qquad (n>0) \tag{3.7b}$$

where $K_n(0) \equiv 0$. As defined, the $K_n(x) \sim O(n^{-2})$ as $n \to \infty$ for any fixed x, and therefore, satisfy (2.54). Equation (3.6) thus becomes

$$\sum_{n=-\infty}^{\infty} |n| a_n [1 + K_{|n|}(k\mathbf{a})] e^{jn\phi} = (k\mathbf{a})^2 [1 + K_0(k\mathbf{a})] a_0 + \frac{(k\mathbf{a})^2 \pi \mathbb{E}(\phi)}{Z_0}, \qquad \phi \in I(\Gamma). \tag{3.8}$$

On the other hand, continuity of H_z across the apertures and the Wronskian relationship

$$J'_{|n|}(k\mathbf{a}) H_{|n|}(k\mathbf{a}) - J_{|n|}(k\mathbf{a}) H'_{|n|}(k\mathbf{a}) = \frac{2j}{\pi k\mathbf{a}} \tag{3.9}$$

give

$$\sum_{n=-\infty}^{\infty} a_n e^{jn\phi} = 0, \qquad \phi \in I(L). \tag{3.10}$$

Defining the quantities

$$\chi_m = K_{|m|}(k\mathbf{a}), \tag{3.11a}$$

$$\xi = (k\mathbf{a})^2 [1 + \chi_0(k\mathbf{a})], \tag{3.11b}$$

$$f(\phi) = \frac{(k\mathbf{a})^2 \pi}{Z_0} \mathbb{E}(\phi), \tag{3.11c}$$

(3.8) and (3.10) constitute the n-series problem:

$$\sum_{m=-\infty}^{\infty} a_m e^{jm\phi} = 0, \qquad \phi \in I(L), \tag{3.12a}$$

$$\sum_{m=-\infty}^{\infty} a_m |m| \tau_m e^{jm\phi} = \xi a_0 + f(\phi), \qquad \phi \in I(\Gamma). \tag{3.12b}$$

It is clearly seen that (3.12) coincides with the $\varepsilon_m = +1$ case of (2.51). Consequently, the unknown amplitudes $a_m (m = 0, \pm 1, \cdots)$ are obtained from the solution system (2.62). The *currents induced* on the cylinder then follow immediately from (3.4a), (3.4b) and (3.9) as

$$J_\phi(\mathbf{a}, \phi) = H^s_{z<}(\mathbf{a}, \phi) - H^s_{z>}(\mathbf{a}, \phi) = \frac{2A_0}{j\pi k\mathbf{a}} \sum_{m=-\infty}^{\infty} a_m e^{jm\phi}. \tag{3.13}$$

The complementary problem, an E-polarized plane wave incident upon a circular cylinder, the metal and apertures now coinciding with L and Γ, respectively, has an analogous solution. Only the E_z, H_ρ and H_ϕ components of the field are excited, the tangential components being related as

$$H_\phi = \frac{-j}{\omega\mu}\partial_\rho E_z. \tag{3.14}$$

The Fourier expansions of the incident and scattered fields are now:

$$E_z^{\text{inc}} = A_0 \sum_{n=-\infty}^{\infty} \left[j^{|n|} J_{|n|}(k\rho) e^{-jn\phi^{\text{inc}}} \right] e^{jn\phi}, \tag{3.15a}$$

$$E_{z>}^s = A_0 \sum_{n=-\infty}^{\infty} c_n J_{|n|}(k\mathbf{a}) H_{|n|}(k\rho) e^{jn\phi} \qquad (\rho > \mathbf{a}), \tag{3.15b}$$

$$E_{z<}^s = A_0 \sum_{n=-\infty}^{\infty} c_n J_{|n|}(k\rho) H_{|n|}(k\mathbf{a}) e^{jn\phi} \qquad (\rho < \mathbf{a}), \tag{3.15c}$$

$$H_{\phi>}^s = -jY_0 A_0 \sum_{n=-\infty}^{\infty} c_n J_{|n|}(k\mathbf{a}) H'_{|n|}(k\rho) e^{jn\phi} \qquad (\rho > \mathbf{a}), \tag{3.15d}$$

$$H_{\phi<}^s = -jY_0 A_0 \sum_{n=-\infty}^{\infty} c_n J'_{|n|}(k\rho) H_{|n|}(k\mathbf{a}) e^{jn\phi} \qquad (\rho < \mathbf{a}), \tag{3.15e}$$

where $Y_0 = k/\omega\mu$ is the free space admittance. Continuity of H_ϕ across the apertures and the Wronskian relation (3.9) give

$$\sum_{n=-\infty}^{\infty} c_n e^{jn\phi} = 0, \qquad \phi \in I(\Gamma). \tag{3.16}$$

Furthermore, satisfaction of the boundary condition $E_z^s(\mathbf{a},\phi) = -E^{\text{inc}}(\mathbf{a},\phi)$ yields

$$\sum_{n=-\infty}^{\infty} c_n J_{|n|}(k\mathbf{a}) H_{|n|}(k\mathbf{a}) e^{jn\phi} = \tilde{\mathbb{E}}(\phi), \qquad \phi \in I(L), \tag{3.17a}$$

where

$$\tilde{\mathbb{E}}(\phi) = \sum_{n=-\infty}^{\infty} \tilde{\mathbb{E}}_n(\phi) e^{jn\phi} = \sum_{n=-\infty}^{\infty} \left[-j^{|n|} J_{|n|}(k\mathbf{a}) e^{-jn\phi^{\text{inc}}} \right] e^{jn\phi}. \tag{3.17b}$$

However, in contrast to the H-polarized case, the dc components of the field are properly extracted by introducing the functions $\tilde{K}_m(x)$ so that

$$\left[j\pi J_0(x) H_0(x) \right]^{-1} = \tilde{K}_0(x), \tag{3.18a}$$

$$\left[-j\pi J_m(x) H_m(x) \right]^{-1} = m\left(1 + \tilde{K}_m(x)\right), \qquad m > 0, \tag{3.18b}$$

where $\tilde{K}_m(0) = 0$. This choice is made to account for the logarithmic singularity of H_0 near $x = 0$. Furthermore, (2.54) is satisfied since $\tilde{K}_m(x) \sim O(m^{-2})$ as $m \to \infty$. Defining the coefficients

$$b_n = c_n J_{|n|}(k\mathbf{a}) H_{|n|}(k\mathbf{a}) - \tilde{\mathbb{E}}_n, \tag{3.19a}$$

$$\tilde{\tau}_n = (1 + \tilde{\chi}_n) \equiv 1 + \tilde{K}_{|n|}(k\mathbf{a}), \tag{3.19b}$$

$$\tilde{\xi} = \tilde{K}_0(k\mathbf{a}), \tag{3.19c}$$

the n-series system defined by (3.16) and (3.17) reduces to the form

$$\sum_{m=-\infty}^{\infty} b_m e^{jm\phi}=0, \qquad \phi\in I(L), \tag{3.20a}$$

$$\sum_{m=-\infty}^{\infty} b_m|m|\tilde{\tau}_m e^{jm\phi}=\tilde{\xi} b_0+\tilde{f}(\phi), \qquad \phi\in I(\Gamma), \tag{3.20b}$$

where

$$\tilde{f}(\phi)=\tilde{\xi}\tilde{\mathbb{E}}_0-\sum_{n\neq 0}|n|\tilde{\mathbb{E}}_n\tilde{\tau}_n e^{jn\phi} \tag{3.21a}$$

$$=\sum_{n=-\infty}^{\infty}\frac{\tilde{\mathbb{E}}_n e^{jn\phi}}{\left[j\pi J_{|n|}(k\mathbf{a})H_{|n|}(k\mathbf{a})\right]}. \tag{3.21b}$$

Therefore, since this system is of the same type as (3.12), its solution system also follows from (2.52), and hence has the same form as the one found in the original H-polarized problem. On the other hand, the currents on the cylinder are now defined as

$$J_z(\mathbf{a},\phi)=H^s_{\phi>}(\mathbf{a},\phi)-H^s_{\phi<}(\mathbf{a},\phi)=\frac{2Y_0A_0}{\pi k\mathbf{a}}\sum_{n=-\infty}^{\infty} c_n e^{jn\phi}. \tag{3.22}$$

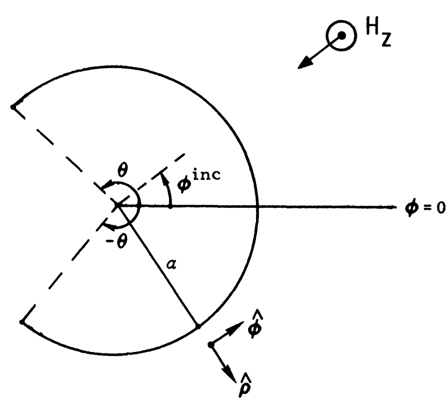

FIG. 1. *Configuration of the scattering of an H-polarized plane wave from a cylinder with an infinite axial slot.*

To illustrate the calculation of the induced currents, consider an H-polarized plane wave coupling to a circular cylinder with a single axial slot. The geometry of this problem is shown in Fig. 1. Equations (3.12) reduce in this case to the dual series equations:

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} a_m e^{jm\phi}=0, && \phi\in(\theta,2\pi-\theta),\\ &\sum_{m=-\infty}^{\infty} a_m|m|\tau_m e^{jm\phi}=\xi a_0+f(\phi), && \phi\in(-\theta,\theta). \end{aligned} \tag{3.23}$$

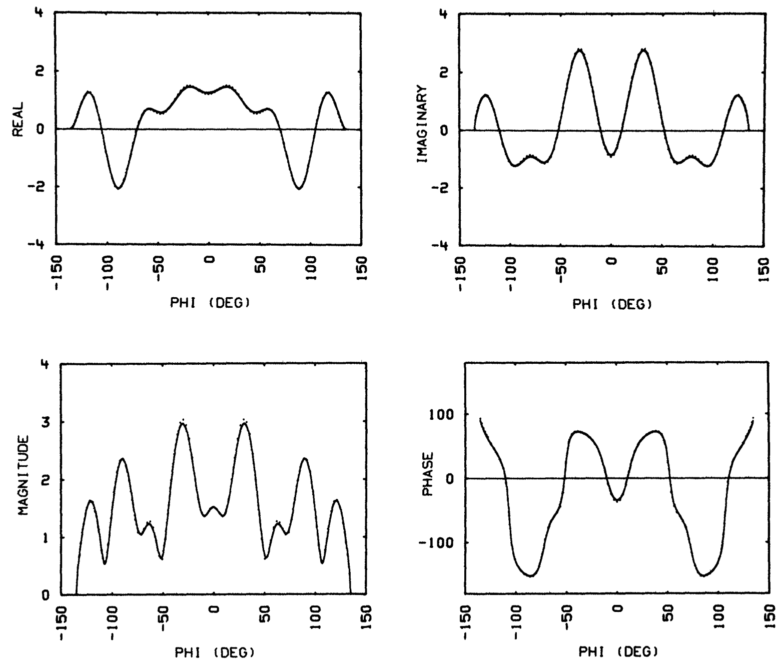

FIG. 2. *Currents calculated by the dual series (----) and the method of moments (····) for an H-polarized plane wave incident at* $\phi^{\text{inc}} = 180°$ *on a cylinder of radius* 1.0 λ *with an aperture angle* $\theta_{ap} = \pi - \theta = 45°$.

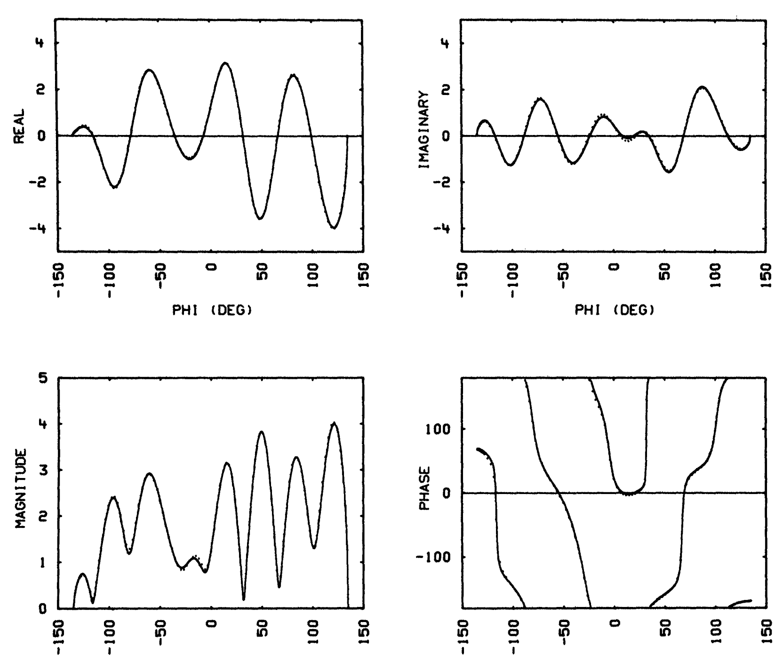

FIG. 3. *Currents calculated by the dual series (----) and the method of moments (····) for an H-polarized plane wave incident at* $\phi^{\text{inc}} = 135°$ *on a cylinder of radius* 1.0 λ *with an aperture angle* $\theta_{ap} = \pi - \theta = 45°$.

These dual series equations have the solution system

$$(3.24)\quad \begin{cases} x_m = ma_m = \xi V_m^0 a_0 + \sum_{n=-\infty}^{\infty} f_n V_m^n - \sum_{n=-\infty}^{\infty} \frac{|n|}{n} \chi_n x_n V_m^n + 2c_0 R_m & (m \neq 0), \\ 0 = \xi V_0^0 a_0 + \sum_{n=-\infty}^{\infty} f_n V_0^n - \sum_{n=-\infty}^{\infty} \frac{|n|}{n} \chi_n x_n V_m^n + 2c_0 R_0 & (m = 0), \\ 0 = (1 + \xi W_0) a_0 + \sum_{n=-\infty}^{\infty} f_n W^n - \sum_{n=-\infty}^{\infty} \frac{|n|}{n} \chi_n x_n W^n + 2c_0 S & (\psi = \pi), \end{cases}$$

where $S_0^0 = S$ and

$$(3.25)\quad f_n = -j^{|n|+1}(k\mathbf{a})^2 \pi J'_{|n|}(k\mathbf{a}) e^{-jn\phi^{\text{inc}}}.$$

The coefficients V_m^n, R_m, W^n and S are given explicitly in [5]. They are combinations of Legendre functions and are readily computed. It has been found [5] that truncating f_n and χ_n in (3.24) for $|n|$ greater than some large value N and using Gauss elimination to solve the remaining finite system yields good numerical approximations for the coefficients c, a_0, $x_{\pm 1}, \cdots, x_{\pm N}$. The remaining coefficients, x_m, for $N < |m| \leqq M$ are given by the expression

$$(3.26)\quad x_m = ma_m = \xi V_m^0 a_0 + \sum_{n=-N}^{N} f_n V_m^n - \sum_{n=-N}^{N} |n| a_n \chi_n V_m^n + 2c_0 R_m.$$

As N approaches ∞, this solution scheme becomes exact. The rate of convergence of the current sum (3.13) is then enhanced by handling the edge behavior analytically. In particular set

$$(3.27)\quad \sum_{m=-\infty}^{\infty} a_m e^{jm\phi} = a_0 + \sum_{m \neq 0} \left(\frac{x_m - \tilde{x}_m}{m} \right) e^{jm\phi} + \sum_{m \neq 0} \frac{\tilde{x}_m}{m} e^{jm\phi}$$

where the term $\tilde{x}_m$ is a large m approximation of x_m. The first sum on the right-hand side of (3.27) is rapidly converging. The second sum is obtained analytically (see [5] for the details); it contains the singular component of the current near an edge of the aperture.

Currents generated with this dual series scheme (solid lines) and with a two-dimensional method of moments code (dotted lines) are shown in Figs. 2 and 3. In Fig. 2, the angle of incidence $\phi^{\text{inc}} = 180°$; in Fig. 3, $\phi^{\text{inc}} = 135°$. The radius of the cylinder in terms of wave length $(\mathbf{a}/\lambda) = 1.0$ and the aperture angle $\theta_{ap} = \pi - \theta = 45°$ in both cases. Moreover, the truncation numbers were chosen to be large: $N = 25$ and $M = 190$, to guarantee the accuracy of the dual series results. Note that both figures demonstrate that the dual series solution readily models the singular behavior of the fields near the edges of the aperture. Furthermore, as discussed in [5], the dual series solution has revealed that the moment method solution will properly describe the current (especially in the shadow region) only if a nonuniform gridding that is finer near aperture edges is employed. The slight inaccuracy of the moment method solution present in both figures disappears when finer gridding is utilized.

4. Comments. The description of coupling to more complex structures such as slitted parabolic or elliptic cylinders leads to the more general n-series problem (2.65). As noted in §3, the structure is assumed to lie on a constant coordinate surface, and the incident and the interior and exterior scattered fields are expanded in the eigenmodes corresponding to that geometry. For instance, for a two-dimensional elliptic cylinder the fields would be expanded in terms of modified and periodic Mathieu functions. The n-series problem follows from enforcing the electromagnetic boundary conditions over the aperture and the metal.

The terminology "n-series problem" needs to be clarified since it is confusing to discover that in general one has a system of $2(n-1)$ equations for an n-series problem. For a single slit $n=2$ and a dual series equation system is obtained which agrees with the notation. On the other hand, for two slits $n=3$ and a system of four equations is obtained in general. However, assuming that the metal-aperture configuration is symmetric about the $\pi=0$ axis, only a triple series equation system need be treated. These symmetric problems are the only ones that have been treated in the past, for example, in [6]. The present approach is not restricted to problems of this type. Nonetheless, the terminology and the subsequent inconvenient notations were chosen so that they reduced to the standard ones encountered in dual and triple series problems.

Note that the Riemann–Hilbert results also explicitly contain, in addition to the correct edge behavior, the multipole behavior of the static solution of infinity. For instance, for a single slit case the dual series system leads to a solution (2.20) which has the limit $\lim_{|z|\to\infty} x(z) \sim c_0/z$. (In fact, since one also has from (2.5) that $\lim_{|z|\to\infty} x(z) \sim x_{-1}/z$, $c_0 \equiv x_{-1}$ in that case.) This indicates that at infinity, the static solution for the slitted cylinder behaves like a line charge or monopole. Similarly, for a cylinder with 2 slits (2.20) has the limit $\lim_{|z|\to\infty} x(z) \sim c_0 z^{-2} + c_1 z^{-1}$. Thus, the static solution contains a dipole as well as a monopole component at infinity.

N-series equations and their solution with Riemann–Hilbert techniques provide an effective approach to a large class of mixed boundary value problems. For instance, this generalized n-series approach generates analytic descriptions of the coupling of electromagnetic waves through apertures into open or enclosed regions. This was illustrated succinctly with the circular cylinder examples. The coupling to a circular cylinder with two axial slits and to a thin spherical shell with a circular aperture are currently under investigation with this method. The analytic solutions to such canonical problems are particularly useful because they are leading to the development of engineering "rules of thumb" for coupling to more general structures. Furthermore, they establish a standard to which large numerical coupling codes can be compared.

5. Appendix: the Riemann–Hilbert problem. Suppose that one is given a simple closed, smooth curve Γ dividing the complex plane into two open sets, the (bounded) interior S_+ and the exterior S_- and two Hölder continuous functions of position on that contour, $T(\gamma)$ and $F(\gamma)$, $T(\gamma)$ being nonvanishing. Let $x(z)$ be a sectionally analytic function, i.e., over the domains S_+ and S_- let $x(z)$ equal, respectively, the analytic functions $x_+(z)$ and $x_-(z)$. Then the Cauchy integral

$$x(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)\,d\zeta}{\zeta - z} \tag{A.1}$$

solves the problem: Find a piecewise analytic function $x(z)$ vanishing at infinity that satisfies on Γ the prescribed transition condition

$$x_+(\gamma) - x_-(\gamma) = F(\gamma), \qquad \gamma \in \Gamma. \tag{A.2}$$

Note that on Γ, the function (A.1) is defined as a Cauchy principal value and satisfies a Hölder condition of the same type as F and the Plemelj–Sokhotskii conditions:

$$x_+(\gamma)=x(\gamma)+\frac{1}{2}F(\gamma), \tag{A.3a}$$

$$x_-(\gamma)=x(\gamma)-\frac{1}{2}F(\gamma). \tag{A.3b}$$

Moreover, the additional condition $x_-(\infty)=0$ can be modified. For instance, if $x(z)$ has a pole of order n at $z=\infty$, the solution of (A.2) is

$$x(z)=\frac{1}{2\pi i}\int_\Gamma \frac{F(\zeta)\,d\zeta}{\zeta-z}+P_n(z), \tag{A.4}$$

where $P_n(z)$ is a polynomial of order n in z, $P_0(z)$ being a constant.

The Riemann–Hilbert problem is a generalization of this problem. In particular, it is desired to find the sectionally analytic function $x(z)$ which satisfies on the contour Γ either the transition condition

$$x_+(\gamma)=T(\gamma)x_-(\gamma) \qquad \text{(homogeneous problem)}, \tag{A.5}$$

or

$$x_+(\gamma)=T(\gamma)x_-(\gamma)+F(\gamma) \qquad \text{(inhomogeneous problem)}. \tag{A.6}$$

A further extension of this problem to open curves and discontinuous coefficients is possible. Note that by generating a solution, $y(z)$, of the homogeneous problem (A.5):

$$y_+(\gamma)=T(\gamma)y_-(\gamma), \tag{A.7}$$

and defining the functions $\Phi=x/y$ and $\Psi=F/y_+$, the inhomogeneous problem (A.6) is reduced to the problem (A.2):

$$\Phi_+(\gamma)-\Phi_-(\gamma)=\Psi(\gamma). \tag{A.8}$$

Acknowledgments. The author wishes to thank J. B. Grant for his computational assistance and K. F. Casey for many enjoyable discussions on this topic. He especially wishes to acknowledge the many astute observations and valuable suggestions made by William A. Johnson during the course of this work. They greatly aided its completion and were very much appreciated.

REFERENCES

[1] H. A. BETHE, *Theory of diffraction by small holes*, Phys. Rev., 66 (1944), pp. 163–182.

[2] K. F. CASEY, *The natural frequencies of the axisymmetric modes of a hollow conducting sphere with a circular aperture*, National Radio Science Meeting, Los Angeles, 1981.

[3] ———, *Capacitive iris in a rectangular waveguide: an exact solution*, UCRL-88377, Lawrence Livermore National Laboratory, Livermore, CA, 1983.

[4] ———, *On the inductive iris in a rectangular waveguide*, UCRL-88308, Lawrence Livermore National Laboratory, Livermore, CA, 1983.

[5] W. A. JOHNSON AND R. W. ZIOLKOWSKI, *The coupling of an H-polarized plane wave to an infinite circular cylinder with an infinite axial slot: a dual series approach*, Radio Sci., 19 (1984), pp. 275–291.

[6] I. N. SNEDDON, *Mixed Boundary Valve Problems in Potential Theory*, North-Holland, Amsterdam, 1966.

[7] Z. S. Agranovich, V. A. Marchenko and V. P. Shestopalov, *The diffraction of electromagnetic waves from plane metallic lattices*, Zh. Tekhn. Fiz., 32 (1962), pp. 381–394; Sov. Phys.- Tech. Phys., 7 (1962), pp. 277–286.

[8] V. N. Koshparënok and V. P. Shestopalov, *Diffraction of a plane electromagnetic wave by a circular cylinder with a longitudinal slot*, Zh. Vychisl. Mat. i Mat. Fiz., 11 (1971), pp. 719–737; U.S.S.R. Comp. Math. and Math. Phys., 11 (1971), pp. 222–243

[9] F. D. Gakhov, *Boundary Value Problems*, Pergamon Press, New York, 1966.