On the Feedback Capacity of $K$-user Cyclic Interference Channel

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Abstract—The $K$-user cyclic interference channel models a situation in which the $k$-th transmitter causes interference only to the $(k-1) \mod K$th receiver. The impact of noiseless feedback on the capacity of this channel is studied by focusing on the linear shift deterministic cyclic interference channel (LD-CIC). For the case of symmetric channel parameters, the symmetric feedback capacity is characterized for all interference regimes. It is shown that the symmetric feedback capacity is in general a function of $K$, the number of users. Furthermore, the capacity gain obtained due to feedback decreases as $K$ increases.

I. INTRODUCTION

Managing the effects of interference is a key issue in currently deployed wireless networks. Among several ways to mitigate or perhaps constructively using interference is to make use of cooperation amongst interfering users. In this paper, we focus on one such issue by studying the impact of noiseless receiver-to-transmitter feedback on the capacity of the $K$-user cyclic interference channel (C-IC). In this model, $K$ transmitters intend to transmit independent messages to $K$ respective receivers and the $k$th transmitter causes interference to the $(k-1) \mod K$th receiver. In addition, we assume the presence of noiseless, causal feedback from the $k$th transmitter to the $k$th receiver. For $K = 2$, this model reduces to the conventional 2-user interference channel. For $K > 2$, this model is a special case of the general $K$-user interference channel (see Figure 1).

The $K$-user cyclic Gaussian interference channel without feedback was recently investigated in [1], where it was shown that the generalized degree-of-freedom of the symmetric $K$-user CIC is the same as that for the 2-user CIC. By a novel generalization of the results of Etkin, Tse and Wang [2], the approximate symmetric capacity was characterized for the weak interference regime and the exact capacity region was characterized for the strong interference regime. A simpler variation of the Gaussian $K$-user CIC has been studied in [3], where the results of [1] are strengthened for the 3-user case. It is shown in [3] that a generalization of the Han-Kobayashi scheme can achieve sum-capacity for some interference regimes.

The 2-user interference channel with various forms of feedback have been investigated recently. Feedback coding schemes for $K$-user Gaussian interference networks have been developed by Kramer in [5]. Outer bounds for the 2-user interference channel with generalized feedback have been derived in [6], [7] (also see references therein).

The 2-user Gaussian interference channel with noiseless feedback was considered in [8] and the feedback capacity region was characterized to within two bits. One of the main findings in [8] is that feedback provides multiplicative gain at high signal-to-noise ratio (SNR) and the gain becomes arbitrary large for certain channel parameters. The key insights that led to this result were obtained by characterizing the feedback capacity region of the linear deterministic (LD) 2-user interference channel. The linear deterministic model despite its simplicity can provide valuable insights for the Gaussian channel model. With this eventual goal in mind, we focus on the linear deterministic $K$-user CIC in this paper.

For the special case of symmetric channel parameters, the linear deterministic IC can be described by a pair of numbers $(n, m)$, where $n$ denotes the number of direct signal levels and $m$ denotes the number of interference levels. The normalized symmetric capacity for $K = 2$ is defined as the maximum value of $R/n$ such that the rate pair $(R, R)$ is achievable. The normalized symmetric capacities for $K = 2$ with and without feedback can be characterized as a function of the interference parameter $\alpha$, defined as $\alpha \triangleq m/n$.

The normalized symmetric capacity without feedback was characterized in [2] and is given as $\min(\max(1 - \alpha, \alpha), 1 - \alpha)$. For the case of symmetric channel parameters, the symmetric feedback capacity is characterized for all interference regimes. It is shown that the symmetric feedback capacity is in general a function of $K$, the number of users. Furthermore, the capacity gain obtained due to feedback decreases as $K$ increases.
of sequences of \( \alpha/2 \). The normalized symmetric feedback capacity for \( K = 2 \) was characterized in [8] and is given as \( \max(1-\alpha/2, \alpha/2) \).

From the results of [1], it is clear that the normalized symmetric capacity without feedback for the \( K \)-user CIC is the same for all \( K \geq 2 \), i.e., it is independent of \( K \), the number of users. It is natural to ask the question that does the symmetric feedback capacity for the \( K \)-user CIC same as that of the 2-user IC? We answer this question in the negative by showing that the normalized symmetric feedback capacity of the \( K \)-user LD-CIC is in general a function of \( K \).

Of particular interest is the very strong interference regime, corresponding to \( \alpha \geq 2 \). In this regime the normalized symmetric feedback capacity for the 2-user case is given by \( \alpha/2 \). This implies that the feedback gain can be unbounded as \( \alpha \) increases. In this regime, we show that the normalized symmetric feedback capacity of the \( K \)-user CIC is given as \( 1 + \frac{\alpha - 2}{K} \).

This result shows that for a fixed \( \alpha \), as the number of users increases, the feedback gain decreases and completely vanishes in the limit \( K \to \infty \).

We characterize the normalized symmetric feedback capacity of the \( K \)-user LD-CIC by deriving novel sum-rate upper bounds and corresponding feedback coding schemes that match the upper bounds. The outer bounds derived in this paper can be regarded as genie aided bounds derived for the 2-user case considered in [8]. However, as \( K \), the number of users increase, selecting appropriate genies becomes prohibitively complex. In particular, for the \( K \)-user CIC, we have a total of \( K! \) sum-rate upper bounds. Depending on the interference parameter \( \alpha \), we carefully select the best upper bound among the \( K! \) upper bounds.

This paper is organized as follows. In Section II, we describe the \( K \)-user cyclic interference channel with feedback. In Section III we describe our main results for the linear deterministic CIC with feedback. We provide the intuition as to why the feedback gain decreases as the number of users increases. In Section IV we present feedback coding schemes for the \( K \)-user LD-CIC. New sum-rate upper bounds are derived for the general \( K \)-user CIC with feedback in Section V. Conclusions and directions for future work are discussed in Section VI.

II. \( K \)-USER CYCLIC INTERFERENCE CHANNEL WITH FEEDBACK

The \( K \)-user cyclic interference channel is described by \( K \) conditional probabilities \( \{p(y_1|x_1, x_2), p(y_2|x_2, x_3), \ldots, p(y_K|x_K, x_1)\} \). An \((T, M_1, \ldots, M_K)\) feedback code for the CIC consists of sequences of \( K \) encoding functions

\[
f_{k,t} : \{1, \ldots, M_k\} \times \mathcal{Y}_k^{t-1} \to X_{k,t}, \quad k = 1, \ldots, K, \quad (1)
\]

and \( K \) decoding functions

\[
g_k : \mathcal{Y}_k^T \to \{1, \ldots, M_k\}, \quad k = 1, \ldots, K. \quad (2)
\]

for \( t = 1, \ldots, T \). The probability of decoding error at decoder \( k \) is denoted by \( P_k \) and is defined as

\[
P_k(g_k(Y_k^T) \neq M_k).
\]

A rate \( K \)-tuple \((R_1, \ldots, R_K)\) is achievable for the \( K \)-user CIC if there exists a \((T, M_1, \ldots, M_K)\) feedback code such that \( \log(M_k)/T \leq R_k - \epsilon_{k,T} \) and \( P_k \leq \epsilon_{k,T} \), where \( \epsilon_{k,T} \to 0 \) as \( T \to \infty \) for all \( k \). The feedback capacity region \( \mathcal{C}_{FB}(K) \) is the set of all achievable \( K \)-tuples.

In this paper, we focus on the symmetric feedback capacity of the \( K \)-user CIC, denoted by \( \mathcal{C}_{FB}^S(K) \), which is defined as the maximum \( R \) such that \((R, \ldots, R) \in \mathcal{C}_{FB}(K)\).

A. Linear deterministic CIC with Feedback

The symmetric linear deterministic CIC is described by a pair of integers \((n, m)\), where \( n \) denotes the number of signal (direct levels) and \( m \) denotes the number of interference levels.

The channel input of transmitter \( k \), denoted by \( X_k \), for \( k = 1, \ldots, K \) is assumed to be of length \( \max(n, m) \).

When \( n \geq m \), we denote

\[
U_k : \text{top-most} \ (n - m) \ \text{bits of} \ X_k
\]
\[
V_k : \text{top-most} \ m \ \text{bits of} \ X_k
\]
\[
L_k : \text{lower-most} \ m \ \text{bits of} \ X_k.
\]

With this notation at hand, we can write the channel outputs for the \( K \)-user LD-CIC as follows:

\[
Y_k = (U_k, L_k \oplus V_{(k+1) \mod (K)}),
\]

for \( k = 1, \ldots, K \).

When \( n < m \), we denote

\[
U_k : \text{top-most} \ (m - n) \ \text{bits of} \ X_k
\]
\[
V_k : \text{top-most} \ n \ \text{bits of} \ X_k
\]
\[
L_k : \text{lower-most} \ n \ \text{bits of} \ X_k.
\]

With this notation at hand, we can write the channel outputs for \( K \)-user LD-CIC as follows:

\[
Y_k = (U_{(k+1) \mod (K)}, L_{(k+1) \mod (K)} \oplus V_k),
\]

for \( k = 1, \ldots, K \).

For instance, when \( n \geq m \), the 3-user LD-CIC is described by the following input-output relationships:

\[
Y_1 = (U_1, L_1 \oplus V_2)
\]
\[
Y_2 = (U_2, L_2 \oplus V_3)
\]
\[
Y_3 = (U_3, L_3 \oplus V_1).
\]

III. MAIN RESULTS

The results for the symmetric feedback capacity presented in this section are described in terms of the interference parameter \( \alpha \), which is defined as the ratio of the number of interference levels to the number of signal levels, i.e.,

\[
\alpha \triangleq \frac{m}{n}.
\]

For notational simplicity, we focus on the normalized symmetric feedback/no-feedback capacities of the LD-CIC re-
respectively, which are defined as follows:

\[ C_{FB}(K) = \frac{C_{FB}^S(K)}{n}, \quad C_{No-FB}(K) = \frac{C_{No-FB}^S(K)}{n}. \]

Before presenting our main results, we first recall previously known results for the LD-CIC. The normalized no-feedback symmetric capacity of the LD-CIC is given as [1]

\[ C_{No-FB}^s(K) = \begin{cases} 
(1 - \alpha), & 0 \leq \alpha \leq 1/2 \\
\alpha, & 1/2 \leq \alpha \leq 2/3 \\
1 - \frac{\alpha}{2}, & 2/3 \leq \alpha \leq 1 \\
\frac{\alpha}{2}, & 1 \leq \alpha \leq 2 \\
1, & \alpha \geq 2.
\end{cases} \tag{8} \]

Note that \( C_{No-FB}^s(K) \) is independent of \( K \), i.e. \( C_{No-FB}^s(K) = C_{No-FB}^s(2) \), for all \( K \). This implies that from the point of view of the symmetric capacity, the behavior of the \( K \)-user system is similar to the \( K = 2 \) user system in the absence of feedback.

The normalized symmetric feedback capacity for \( K = 2 \) is given as [8]

\[ C_{FB}^s(2) = \begin{cases} 
1 - \frac{\alpha}{2}, & \alpha \leq 1 \\
\alpha, & \alpha \geq 1.
\end{cases} \tag{9} \]

In the light of above observations, it is natural to ask that does the behavior of \( K \)-user LD-CIC mimic the behavior of the \( K = 2 \) system in presence of feedback. We answer this question in the negative by showing that the normalized symmetric feedback capacity for \( K > 2 \) is in general a function of \( K \). Moreover, the normalized symmetric feedback capacity of \( K = 2 \) always serves as an upper bound for the normalized symmetric feedback capacity for a LD-CIC with \( K > 2 \) users.

The main results of this paper are described in the following theorem:

**Theorem 1:** The normalized symmetric feedback capacity, \( C_F^s(K) \) of the \( K \)-user LD-CIC satisfies

\[ C_{FB}^s(K) = \begin{cases} 
(1 - \alpha) + \frac{\alpha}{K}, & 0 \leq \alpha \leq 1/2 \\
\alpha + \frac{2 - 3\alpha}{K}, & 1/2 \leq \alpha \leq 2/3 \\
1 - \frac{\alpha}{K}, & 2/3 \leq \alpha \leq 1 \\
\frac{\alpha}{K}, & 1 \leq \alpha \leq 2 \\
1 + \frac{\alpha - 2}{K}, & \alpha \geq 2.
\end{cases} \tag{10} \]

Theorem 1 is proved in two parts: feedback coding schemes are presented in Section IV and corresponding upper bounds for the normalized symmetric feedback capacity are obtained in Section V. In Figure 2, the normalized symmetric feedback capacities are shown for the \( K \)-user LD-CIC, when \( K = 2, 4 \) and \( K = 10 \).

**Remark 1:** Theorem 1 shows that \( C_{FB}^s(K) \) can be strictly less than \( C_{FB}^s(2) \) (see Figure 2). Secondly, it also shows that \( C_{FB}^s(2) \) is monotonically decreasing in \( K \). Hence, as the number of users in the system increase, the capacity gain obtained via feedback decreases. Furthermore, in the limit

\[ F_B\rightarrow \infty, \] the feedback gain vanishes, i.e., we have

\[ \lim_{K\rightarrow \infty} C_{FB}^*(K) = C_{No-FB}^s(2). \tag{11} \]

IV. FEEDBACK CODING SCHEMES

A. Very-Weak interference: \( 0 \leq \alpha \leq 1/2 \)

In this regime, we show that \( K(n-m)+m \) bits per user are achievable in \( K \) channel uses.

As an example, we start with the case in which \( K = 4 \) and \( m = 1, n = 3 \) so that \( \alpha = 1/3 \). To achieve 9 bits per user in 4 channel uses, the following coding scheme is used (see Figure 3):

- In the first channel use, each encoder transmits fresh bits on all levels (e.g. encoder 1 sends a, a2, a3).
- Upon receiving feedback, each encoder can decode the upper most bit of the next encoder (encoder 1 decodes b1, encoder 2 decodes c1).
- In all subsequent channel uses, each encoder transmits the previously decoded bit in the top most level and fresh information bits in the remaining lower two levels (at \( t = 2 \) encoder 1 transmits b1 in the top level and a4, a5 in the two lower levels).
- From Figure 3 it is clear that each user can reliably transmit 9 bits to its decoder in 4 channel uses. Hence, this scheme yields a normalized symmetric rate of \((9/4) \times (1/3) = 3/4\).

This scheme can be readily generalized for arbitrary numbers of users \( K \) and for any \( \alpha \in [0,1/2] \) as follows: at \( t = 1 \), every encoder transmits \( n \) fresh information bits. Using feedback, the \( k \)-th encoder decodes the lower most \( m \) bits transmitted by the \( (k+1) \)th encoder. For all subsequent \( 1 < t \leq K \), the \( k \)-th encoder transmits the previously decoded \( m \) bits on the top most \( m \) levels and transmits fresh information in the lower \( (n-m) \) levels. This scheme achieves \( K(n-m)+m \) bits per user in \( K \) channel uses and the rate achievable is \((n-m)+m/K\). Hence for \( \alpha \in [0,1/2] \), we
all encoders remains silent in the $X$ channel uses. Hence, we have

$$C_{FB}(K) \geq (1 - \alpha) + \frac{\alpha}{K}. \quad (12)$$

**B. Weak interference:** $1/2 \leq \alpha \leq 2/3$

For this regime, we present a feedback coding scheme that achieves $Km + (2n - 3m)$ bits per user in $K$ channel uses. We break the channel input of encoder $k$ into four mutually exclusive levels as follows: $X_k(t) = (X_{k,1}(t), X_{k,2}(t), X_{k,3}(t), X_{k,4}(t))$, where the number of bits in $X_{k,r}(t)$ are $(2m - n), (2n - 3m), (2m - n)$ and $(n - m)$, for $r = 1, 2, 3$ and 4, respectively.

At $t = 1$, each encoder transmits fresh information bits on $X_{k,1}(1), X_{k,2}(1)$ and $X_{k,4}(1)$ levels. For all $t \in \{1, \ldots, K\}$, all encoders remains silent in the $X_{k,3}(t)$ level. At $t$, due to feedback, encoder $k$ can decode the $t$ bits transmitted by the encoder $(k+1) \mod (K)$ in the level 2, i.e., it can decode $X_{(k+1) \mod (K),2}(t)$. For all $1 < t \leq K$, encoder $k$ transmits

$$X_k(t) = (X_{k,1}(t), X_{(k+1) \mod (K),2}(t-1), \phi, X_{k,4}(t))$$

It is clear that $Km$ bits are achievable from the levels 1 and 4. A gain of $(2n - 3m)$ bits is provided by feedback in $K$ usages of the channel. It can be easily verified that this coding scheme yields $Km + (2n - 3m)$ bits per user in $K$ channel uses. Hence, we have

$$C_{FB}(K) \geq \alpha + \frac{(2 - 3\alpha)}{K}. \quad (13)$$

**C. Moderate-strong interference:** $2/3 \leq \alpha \leq 2$

In this regime, Theorem 1 shows that feedback does not increase the normalized symmetric capacity and hence the no-feedback coding scheme in [1] suffices.

**D. Very-strong interference:** $\alpha \geq 2$

In this regime, we will show that $(K - 2)n + m$ bits per user are achievable in $K$ channel uses.

As an example, we start with the case in which $K = 4$ and $m = 3$ and $n = 1$, so that $\alpha = 3$. To achieve 5 bits per user in 4 channel uses, the following coding scheme is used (see Figure 4):

- In all channel uses, each encoder remains silent in the lower most bit. In the first channel use, each encoder transmits 2 fresh bits (for instance, encoder 1 sends $a_1, a_2$ and encoder 2 sends $b_1, b_2$). Using feedback, each encoder can decode the second bit transmitted by the encoder interfering its decoder (encoder 1 decodes $b_2$, encoder 2 decodes $c_2$ etc.).
- In all subsequent channel uses, each encoder transmits a fresh information bit in the top-most level and the previously decoded bit in the second level (for instance, at $t = 2$, encoder 2 sends the fresh bit $b_2$ in the top-most level and the decoded bit $c_2$ in the second level).
- From Figure 4, it is clear that in 4 channel uses, using the top most level, each decoder receives 4 bits. One more bit is received from the interfering user in the final channel use (for instance, the bit $a_2$ is received at decoder 1 in a delayed manner). This scheme yields a rate of $5/4$ per user.

This scheme can be readily generalized for arbitrary numbers of users $K$ and for any $\alpha \geq 2$ as follows: for any $1 \leq t \leq K$, all encoders do not transmit any information in the lower most $n$ levels. At $t = 1$, the $k$th encoder transmits $(m - n)$ fresh information bits in the top $(m - n)$ levels. Using feedback, it decodes the $(m - n)$ bits transmitted by the $(k+1)$ th encoder. For any $1 < t \leq K$, the $k$th encoder transmits fresh information on the top most $n$ levels and in the remaining $(m - 2n)$ levels, it transmits the lower $(m - 2n)$ bits decoded at $(t-1)$. This scheme achieves $Kn + (m - 2n)$ bits per user in $K$ channel uses. Hence for $\alpha \geq 2$, we have

$$C_{FB}(K) \geq 1 + \frac{(\alpha - 2)}{K} \quad (14)$$

From the optimal feedback coding scheme described above, it is clear that the gain obtained via feedback should decrease as the number of users increase. To substantiate this claim, we note that when $\alpha \geq 2$ (corresponding to $m > 2n$), a
normalized per-user rate of 1 can always be achieved without feedback by remaining silent on the lowestmost $n$ levels and sending fresh information in the top-most $n$ levels. However, if feedback, each user can send additional information in the middle $(m-2n)$ levels in the first channel use. This additional information can eventually reach the intended decoder via the delayed feedback path in $K$ channel uses. Therefore, feedback can boost the rate from $Kn$ bits to $Kn+(m-2n)$ bits in $K$ usages of the channel. Therefore, the rate gain obtained via feedback is $(m-2n)/K$ which decreases as $K$ increases.

V. UPPER BOUNDS ON THE FEEDBACK SUM-CAPACITY

In this section, we present two types of upper bounds on the sum-capacity of the $K$-user LD-CIC. The type-I upper bound allows us to show that the normalized symmetric feedback capacity for the $K$-user LD-CIC is always upper bounded by the symmetric feedback capacity of the 2-user system. The type-II upper bound is in fact a set of $K!$ genie-aided upper bounds, where each upper bound corresponds to a permutation of $K$ users. This set of upper bounds are in fact valid for the general cyclic interference channel with feedback, i.e., they are not specific for the linear deterministic model. We present the type-I upper bound in the following theorem:

**Theorem 2:** The normalized symmetric feedback capacity of the $K$-user LD-CIC satisfies

$$C^FB_{\text{sum}}(K) \leq \max \left( 1 - \frac{\alpha}{2}, \frac{\alpha}{2} \right).$$

The proof of Theorem 2 is given in the appendix. The main idea behind this upper bound is to show that

$$R_j + R_{(j+1) \mod (K)} \leq \max(2n-m, m),$$

for $j = 1, \ldots, K$. By adding all such $K$ upper bounds and normalizing by $2nK$, we obtain the desired bound stated in Theorem 2. Theorem 2 along with (8) leads to the conclusion that feedback does not increase the symmetric capacity of the $K$-user LD-CIC in the regime $\alpha \in [2/3, 2]$. We next present the type-II upper bound:

**Theorem 3:** Fix a permutation $\pi = \{\pi_1, \ldots, \pi_K\}$ order, then the feedback sum-capacity of the general $K$-user cyclic interference channel is upper bounded as follows:

$$C^FB_{\text{sum}}(K) \leq \max_{p(x_{1}, \ldots, x_{K})} \sum_{k=1}^{K} H(Y_{\pi_k}|X_{\pi_1}, Y_{\pi_1}, \ldots, X_{\pi_{k-1}}, Y_{\pi_{k-1}}) - H(Y_1, \ldots, Y_K|X_1, \ldots, X_K).$$

To illustrate by an example, consider the case when $K = 3$, for which Theorem 3 yields 6 upper bounds on the feedback sum capacity:

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_1) + H(Y_2|X_1, Y_1) + H(Y_3|X_1, X_2, Y_1, Y_2) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right]$$

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_1) + H(Y_3|X_1, Y_1) + H(Y_2|X_1, X_3, Y_1, Y_3) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right]$$

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_2) + H(Y_1|X_2, Y_2) + H(Y_3|X_1, X_2, Y_1, Y_2) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right]$$

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_2) + H(Y_3|X_2, Y_2) + H(Y_1|X_2, X_3, Y_2, Y_3) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right]$$

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_3) + H(Y_1|X_3, Y_3) + H(Y_2|X_1, X_3, Y_1, Y_3) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right]$$

$$\max_{p(x_{1}, x_{2}, x_{3})} \left[ H(Y_3) + H(Y_2|X_3, Y_3) + H(Y_1|X_2, X_3, Y_2, Y_3) - H(Y_1, Y_2, Y_3|X_1, X_2, X_3) \right].$$
For an arbitrary $K$, Theorem 3 gives a total of $K!$ upper bounds. Optimization of these bounds for the general $K$ user case and asymmetric channel gains become prohibitively complex. For the scope of this paper, we are interested in the case of symmetric linear deterministic CIC. Depending on the range of the interference parameter $\alpha$, we carefully select one of the type-II bounds and evaluate it to obtain the desired converse result as stated in Theorem 1.

A. Weak-to-moderate interference: $0 \leq \alpha \leq 2/3$

In this regime, we select the type-II upper bound corresponding to the following permutation order:

$$\pi = (1, 2, \ldots, K)$$

(17)

Theorem 3 yields the following upper bound on the sum-capacity:

$$C_{FB}^{\text{sum}}(K) \leq \sum_{k=1}^{K} H(Y_k|X^{k-1}, Y^{k-1}) - H(Y_1, \ldots, Y_K|X_1, \ldots, X_K)$$

(18)

$$= \sum_{k=1}^{K} H(Y_k|X^{k-1}, Y^{k-1})$$

(19)

$$= H(Y_1) + H(Y_2|X_1, Y_1)$$

$$+ \ldots + H(Y_K|X_1, Y_1, \ldots, X_{K-1}, Y_{K-1})$$

(20)

$$\leq n + \sum_{k=2}^{K-1} H(Y_k|X_{k-1}, Y_{k-1})$$

$$+ H(Y_K|X_1, Y_1, X_{K-1}, Y_{K-1}),$$

(21)

where (18) follows from the fact that $(Y_1, \ldots, Y_K)$ are all deterministic functions of $(X_1, \ldots, X_K)$, (21) follows from the fact that $H(Y_1) \leq n$.

To further upper bound (21), we first recall the notation used for $n \geq m$ in (3):

$U_k$: top-most $(n - m)$ bits of $X_k$

$V_k$: top-most $m$ bits of $X_k$

$L_k$: lower-most $m$ bits of $X_k$.

For any $2 \leq k \leq (K - 1)$, we have the following sequence of inequalities:

$$H(Y_k|X_{k-1}, Y_{k-1})$$

$$= H(Y_k|X_{k-1}, Y_{k-1}, V_k)$$

$$= H(U_k, L_k \oplus V_{k+1}|X_{k-1}, Y_{k-1}, V_k)$$

(22)

$$\leq H(U_k|V_k) + H(L_k \oplus V_{k+1})$$

(23)

$$\leq \max(0, n - 2m) + m,$$

(24)

where (22) is due to the fact that $V_k$ can be determined from $(X_{k-1}, Y_{k-1})$.

Finally we upper bound the last term in (21) as follows:

$$H(Y_K|X_1, Y_1, X_{K-1}, Y_{K-1})$$

$$= H(Y_K|V_1, X_1, Y_1, X_{K-1}, Y_{K-1}, V_K)$$

$$= H(U_K, L_K \oplus V_1|V_1, X_K, X_1, Y_1, X_{K-1}, Y_{K-1})$$

$$\leq H(U_K, L_K|V_K)$$

$$= H(X_K|V_K)$$

$$\leq (n - m).$$

(25)

(26)

(27)

(28)

(29)

(30)

Using (25) and (30), we can further upper bound (21) to obtain

$$C_{FB}^{\text{sum}}(K) \leq (1 - 2\max(0, n - 2m) + m) + (n - m).$$

(31)

Therefore, the normalized symmetric feedback capacity is upper bounded as follows:

$$C_{FB}^{\text{sym}}(K) \leq \max(\alpha, 1 - \alpha) + \frac{\max(\alpha, 2 - 3\alpha)}{K},$$

(32)

which can also be written as

$$C_{FB}^{\text{sym}}(K) \leq \left\{ \begin{array}{ll}
(1 - \alpha) + \frac{\alpha}{K}, & 0 \leq \alpha \leq 1/2 \\
\alpha + \frac{2 - 3\alpha}{K}, & 1/2 \leq \alpha \leq 2/3.
\end{array} \right.$$$$

(33)

Note that the upper bound alone shows that in the limit $K \to \infty$ the upper bound converges to the no-feedback symmetric capacity. This implies that in the limit of large $K$, the feedback gain vanishes to zero.

B. Very strong interference: $\alpha \geq 2$

In this regime, we select the type-II upper bound corresponding to the following permutation order:

$$\pi = (1, K, K - 1, K - 2, \ldots, 3, 2).$$

(34)

Theorem 3 yields the following upper bound on the sum-capacity:

$$C_{FB}^{\text{sum}}(K) \leq H(Y_1) + H(Y_K|X_1, Y_1)$$

$$+ H(Y_K|X_1, X_K, Y_1, Y_K) + \ldots$$

$$+ H(Y_2|X_1, X_3, \ldots, X_K, Y_1, Y_3, \ldots, Y_K)$$

$$\leq H(Y_1) + H(Y_K|X_1, Y_1)$$

$$+ \sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1})$$

$$+ H(Y_2|X_1, Y_1, X_3, Y_3).$$

(35)

(36)

To further upper bound (36), we recall the notation used for $n \geq m$ in (5):

$U_k$: top-most $(m - n)$ bits of $X_k$

$V_k$: top-most $n$ bits of $X_k$

$L_k$: lower-most $n$ bits of $X_k$.

We now upper bound the terms in (36) as follows. We first have the trivial upper bound $H(Y_1) \leq m$. We then bound
the second term in (36) as follows:
\[
H(Y_K|X_1,Y_1) = H(U_1, L_1 + V_K|X_1,Y_1) = H(L_1 + V_K|X_1,Y_1)
\]
\[
\leq n. \tag{38}
\]

Next, for any \(3 \leq k \leq (K-1)\), we have
\[
H(Y_k|X_{k+1}, Y_{k+1}) = H(U_{k+1}, L_{k+1} + V_k|X_{k+1}, Y_{k+1})
\]
\[
= H(L_{k+1} + V_k|X_{k+1}, Y_{k+1}) \leq n. \tag{41}
\]

which implies that
\[
\sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1}) \leq (K-3)n. \tag{43}
\]

Finally, we have
\[
H(Y_2|X_1, Y_1, X_3, Y_3)
\]
\[
= H(U_3, L_3 + V_2|X_1, Y_1, X_3, Y_3) \tag{44}
\]
\[
= H(V_2|X_1, Y_1, X_3, Y_3) \tag{45}
\]
\[
= H(V_2|U_2, X_1, Y_2, X_3, Y_3) \tag{46}
\]
\[
= 0, \tag{47}
\]

where (47) follows from the fact that \(\alpha \geq 2\) corresponds to the case in which \(m - n \geq n\) and therefore \(V_2\) is completely determined by \(U_2\).

Using (39), (43) and (47), we have the following upper bound from (36):
\[
C_{FB}^{FB}(K) \leq H(Y_1) + H(Y_K|X_1,Y_1)
\]
\[
+ \sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1})
\]
\[
+ H(Y_2|X_1, Y_1, X_3, Y_3) \leq m + (K-2)n. \tag{48}
\]

Normalizing this upper bound by \(nK\), we obtain
\[
C_{FB}^{FB}(K) \leq \frac{C_{FB}^{FB}(K)}{nK} \leq \frac{m + (K-2)n}{nK} \leq 1 + \frac{\alpha - 2}{K}, \tag{52}
\]

which is the desired upper bound on the normalized symmetric feedback capacity.

\section{Conclusions}

In this paper, we have considered the \(K\)-user linear deterministic cyclic interference channel with noiseless feedback. The symmetric feedback capacity has been completely characterized for all interference regimes. It has been shown that the capacity gain obtained via feedback decreases as the number of users increases. It has been shown that the normalized symmetric feedback capacity for \(K > 2\) is a skewed “\(V\)" shape. Moreover as \(K \to \infty\), the resulting skewed “\(V\)”-curve converges to the well known “\(W\)” curve corresponding to the no-feedback capacity. We believe that the linear deterministic model should provide useful insights for characterizing the approximate feedback capacity of the Gaussian \(K\)-user cyclic interference channel, which is part of our planned future work.

\section*{Appendix}

\subsection{Proof of Theorem 2}

We show that the normalized symmetric feedback capacity is upper bounded as follows:
\[
C_{FB}^{FB}(K) \leq \max \left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{53}
\]

To prove (53), we first prove the following upper bound on the sum of rates of users 1 and 2.
\[
T(R_1 + R_2)
\]
\[
= H(W_1) + H(W_2) \tag{54}
\]
\[
= H(W_1|W_3, \ldots, W_K) + H(W_2|W_1, W_3, \ldots, W_K) \tag{55}
\]
\[
\leq I(W_1; Y_1^T|W_3, \ldots, W_K) 
\]
\[
+ I(W_2; Y_2^T, Y_3^T|W_1, W_3, \ldots, W_K) + \epsilon_n \tag{56}
\]
\[
= I(W_1; Y_1^T|W_3, \ldots, W_K) 
\]
\[
+ H(Y_2^T, Y_3^T|W_1, W_3, \ldots, W_K) + \epsilon_n \tag{57}
\]
\[
= H(Y_1^T|W_3, \ldots, W_K) 
\]
\[
+ H(Y_2^T|Y_1^T, W_1, W_3, \ldots, W_K) + \epsilon_n \tag{58}
\]
\[
\leq H(Y_1^T) + H(Y_2^T, Y_3^T|W_1, W_3, \ldots, W_K) + \epsilon_n \tag{59}
\]
\[
\leq T \max(m,n) + H(Y_2^T, Y_1^T, W_1, W_3, \ldots, W_K) + \epsilon_n \tag{60}
\]
\[
\leq T \max(m,n) + T|n-m| + \epsilon_n, \tag{61}
\]

where (55) follows from the fact that the messages \(W_1, \ldots, W_K\) are all mutually independent, (56) follows from Fano’s inequality [9], (57) follows from the deterministic nature of the channel model and (59) follows from the fact that conditioning reduces entropy.

Before proving (61) we first prove the following claim:

\textbf{Claim 1:} \((X_{1t}, X_{3t}, \ldots, X_{KT})\) is as deterministic function of \((Y_1^{t-1}, W_1, W_3, \ldots, W_K)\).

\textbf{Proof:} First note that from (1), we have
\[
X_{1t} = f_{1t}(W_1, Y_1^{t-1}), \tag{62}
\]
and
\[
X_{Kt} = f_{Kt}(W_K, Y_K^{t-1}) \tag{63}
\]
\[
= f_{Kt}(W_K, X_1^{t-1}, X_K^{t-1}), \tag{64}
\]

which together imply that
\[
(X_{1t}, X_{Kt}, X_1^{t-1}, X_K^{t-1}) = f(W_1, W_K, Y_1^{t-1}). \tag{65}
\]

Repeating this argument for \(k = K-1, \ldots, 3\), the proof of claim is straightforward.
We now bound the second term in (60) as follows:

\[ H(Y_2^T | Y_1^T, W_1, W_3, \ldots, W_K) \]

\[ \leq \sum_{t=1}^{T} H(Y_{2t} | Y_{1t}, W_1, W_3, \ldots, W_K, Y_1^{t-1}) \]

\[ = \sum_{t=1}^{T} H(Y_{2t} | Y_{1t}, X_{1t}, W_1, X_3t, \ldots, X_{Kt}, W_K, Y_1^{t-1}) \]

\[ \leq \sum_{t=1}^{T} H(Y_{2t} | Y_{1t}, X_{1t}, X_3t, \ldots, X_{Kt}) \]

(67)

\[ \leq T |n - m|, \]

(68)

where (67) follows from Claim 1 and (69) follows from the fact that \((X_{1t}, Y_{1t}, X_3t)\) completely determine at least \(m\) levels of \(Y_{2t}\). This completes the proof of (61). Dividing (61) by \(nT\) and taking the limit \(T \to \infty\), we have \(\epsilon_n \to 0\), which yields

\[ \frac{R_1 + R_2}{n} \leq \max \left( \frac{m}{n} - 1 \right) + \left| 1 - \frac{m}{n} \right| \]

(70)

\[ = \max(\alpha, 1) + |1 - \alpha| \]

(71)

\[ = \max(2 - \alpha, \alpha). \]

(72)

In a similar manner it can be shown that for any \(1 \leq j \leq K\),

\[ \frac{R_j + R_{(j+1) \mod(K)}}{n} \leq \max(2 - \alpha, \alpha). \]

(73)

Adding all such \(K\) upper bounds, we obtain

\[ \frac{2(R_1 + \ldots + R_K)}{n} \leq K \max(2 - \alpha, \alpha), \]

(74)

and hence,

\[ \mathcal{C}_{PB}(K) \leq \max \left( 1 - \frac{\alpha}{2}, \frac{\alpha}{2} \right). \]

(75)

This upper bound on the normalized symmetric feedback capacity is independent of \(K\) and is the same as the normalized symmetric capacity without feedback when \(\alpha \in [2/3, 2]\). Hence, for this interference regime feedback does not increase the symmetric capacity. Also note that the range of \(\alpha\) in deriving these bounds is immaterial and hence from symmetric feedback capacity point of view of, the feedback capacity for \(K = 2\) users always serves as an upper bound for any \(K > 2\).

### B. Proof of Theorem 3

For a permutation order \(\pi = (\pi_1, \pi_2, \ldots, \pi_K)\) for the \(K\) users, we have the following upper bound on the sum-rate:

\[ T \left( \sum_{k=1}^{K} R_{\pi_k} \right) \leq \sum_{k=1}^{K} H(W_{\pi_k}) \]

\[ = \sum_{k=1}^{K} H(W_{\pi_k} | W_{\pi_1}, \ldots, W_{\pi_{k-1}}) \]

(76)

\[ \leq \sum_{k=1}^{K} I(W_{\pi_k}; Y_{\pi_1}^T, \ldots, Y_{\pi_k}^T | W_{\pi_1}, \ldots, W_{\pi_{k-1}}) + \epsilon_n \]

(77)

\[ = \sum_{k=1}^{K} \left[ H(Y_{\pi_1}^T, \ldots, Y_{\pi_k}^T | W_{\pi_1}, \ldots, W_{\pi_{k-1}}) \right. \]

\[ - H(Y_{\pi_1}^T, \ldots, Y_{\pi_k}^T | W_{\pi_1}, \ldots, W_{\pi_{k-1}}, W_k) \]

\[ + \epsilon_n \]

(78)

\[ = \sum_{k=1}^{K} \left[ \sum_{t=1}^{T} H(Y_{\pi_1}^T, \ldots, Y_{\pi_k}^T | W_{\pi_1}, \ldots, W_{\pi_{k-1}}, Y_t, \ldots, Y_{\pi_k}) \right. \]

\[ - H(Y_{\pi_1}, \ldots, Y_{\pi_k} | Y_t, \ldots, Y_{\pi_k}) \]

\[ + \epsilon_n \]

(79)

\[ \leq \sum_{t=1}^{T} \left[ \sum_{k=1}^{K} H(Y_{\pi_1}^T, \ldots, Y_{\pi_k}^T | W_{\pi_1}, \ldots, W_{\pi_{k-1}}, Y_t, \ldots, Y_{\pi_k}) \right. \]

\[ - H(Y_{\pi_1}, \ldots, Y_{\pi_k} | Y_{\pi_1}, \ldots, Y_{\pi_k}, X_{\pi_1}, \ldots, X_{\pi_k}) \]

\[ + \epsilon_n \]

(80)

where (76) follows from the independence of the messages, (77) follows from Fano’s inequality [9], and (79) follows from the fact that the negative term corresponding to the \(k\)th bracket is cancelled by a part of the positive term in the \((k + 1)\)th bracket, for \(k = 1, \ldots, (K - 1)\). Finally, dividing (81) by \(T\) and letting \(T \to \infty\), we have the proof of Theorem 3.

### References


