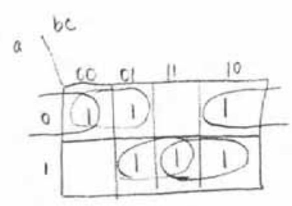
Lecture 10 E

Espresso Expand



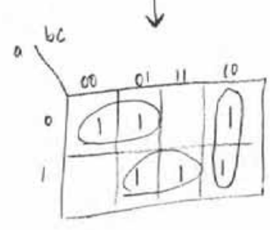
Expand operation goals

1. minimize number of cubes in a cover
2. find prime cubes (dependent on least number of variables)

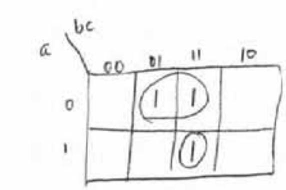


$$\begin{bmatrix} 0 & - & 0 \\ 0 & 0 & - \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}$$

4 to 3 cubes

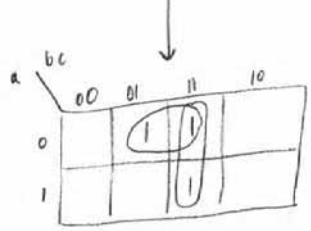


$$\begin{bmatrix} 0 & 0 & - \\ 1 & - & 1 \\ - & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & - & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

5 to 4 literals



$$\begin{bmatrix} 0 & - & 1 \\ - & 1 & 1 \end{bmatrix}$$

## Expand

```
EXPAND(F, R)
begin
    F ← DECREASING_ORDER(F)          // process largest cubes first

    for(j=1, ..., |F|) begin         // process one cube at a time
        (W, F^j) ← EXPAND1(F^j, R, F) // expand to a prime
        F ← (F ∪ {F^j}) – W          // add prime to cover, remove anything covered by prime
    end

    Return F
end
```

Order matters!

F ← DECREASING_ORDER (F)

Rearranges cubes in F so largest cubes processed first, then in order of decreasing size

Why? Larger cubes are more likely to cover others and less likely to be covered by others

Each cube processed separately, trying to expand cube into a prime.

call EXPAND1 heuristic guided by "Blocking Matrix" and "covering matrix" to determine the expanded cube $c^+$

# Blocking Matrix

**Blocking Matrix**

Assuming a single-output function, B is a 0-1 matrix determined by c (cube to be expanded) and R (the off-set). Rows are in one-to-one correspondence with the cubes of $R^i$. The elements of B are defined as follows

$$B_{ij} = \begin{cases} 1 & \text{if } c_j = 1 \text{ and } R_{ij} = 0, \text{ or} \\ & \quad c_j = 0 \text{ and } R_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE 1

$F^i$ = 101
020
002
100

R = 112
211

C = 002 ⟶

$$B = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{matrix}$$

$C_3 = 2 / R_{1,3} = 2 \therefore B = 0$

$C_2 = 0 / R_{2,2} = 1 \therefore B = 1$

$C_1 = 0 / R_{2,1} = 2 \therefore B = 0$



Purpose of B is to ensure expansion of cube c doesn't intersect with the off-set (R)

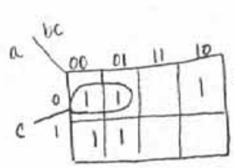
What does a "1" in the blocking matrix mean?

if an entry i,j is "1", then variable j must remain the same (cannot be changed to "2") or cube will conflict with R

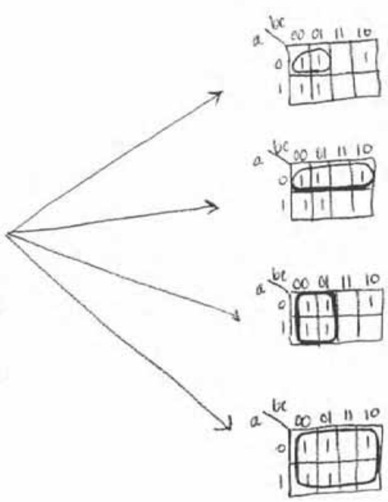
F = 101
020
002
100

R = 112
211

c = 002 ⟶ $\boxed{B = \begin{matrix}110\\010\end{matrix}}$




How can cube c be expanded?



c = 002 → $c^+$ = 002

B = 110
if we keep a and b the same, c can be expanded

c = 002 → $c^+$ = 022 X

C = 002 → $c^+$ = 202

B = 010
if we keep b the same, a and c can be expanded

c = 002 → $c^+$ = 222 X

# Lowering Set

**Lowering Set**

L is the lowering set specifies which columns of c are "lowered" to their original value, thereby defining the expanded cube $c^+$ as follows

$$c^+(L, c)_j = \begin{cases} c_j, & j \in L \\ 2 & \text{otherwise} \end{cases}$$

EXAMPLE 2

$c = 002$ $\quad L = \{2\} \longrightarrow c^+ = 202$ // col 1 and 3 are raised to 2, col 2 remains the same

$c = 101$ $\quad L = \{1,3\} \longrightarrow c^+ = 121$ // col 2 is raised to 2, col 1 and 3 remain the same

$c = 100$ $\quad L = \{1,2,3\} \longrightarrow c^+ = 100$ // col 1,2,3 remain the same



We want L to be as small as possible to obtain the largest $c^+$, but we must ensure the expand cube is valid

$c^+(L,c)$ is an implicant (valid circle) of F if L is a column covering of B

every row of B contains a "1" in some column that appears in L

EXAMPLE 3

$B = 110 \leftarrow$ col 2 in L
$\quad\; 011 \leftarrow$ col 2 in L

$L = \{2\}$ is a valid column covering

$B = 110 \leftarrow 1$ is in L
$\quad\; 011 \leftarrow 3$ is in L

$L = \{1,3\}$ is a valid column covering

$B = 110 \leftarrow 1$ is in L
$\quad\; 011 \leftarrow$ this row not covered by L

$L = \{1\}$ NOT a valid column covering

Number of literals in $c^+$ are determined by the cardinality (# elements) in $L$. If $L$ is a minimum column cover of $B$, then $c^+$ is the largest implicant of $c$

$F = 101$
$\quad 020$
$\quad 002$
$\quad 100$

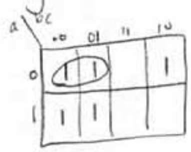
$R = 112$
$\quad 211$

$c = 002$

$B = 110$
$\quad 010$

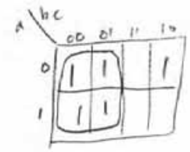
$L = \{1, 2\}$
valid column covering, not minimum column covering

$c^+ = 002$



$L = \{2\}$
valid column covering, minimum column covering

$c^+ = 202$



smaller $|L|$ yields better cover (implicant vs. prime implicant)
Search for minimum cover of $L$ can be guided by a "covering matrix"



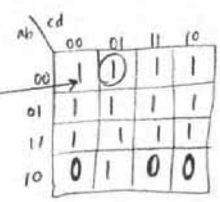
Lowering set seems hard. If blocking matrix tells me how I can expand the cube, why not pick an entry is $B$ with the most "0"s

Example 5

$F = 2122$
$\quad 0001$
$\quad 1001$
$\quad 0012$
$\quad 0020$

$R = 1010$
$\quad 1000$
$\quad 1011$

$c = 0001$

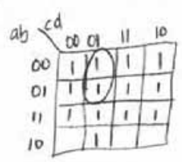


Let's try to expand $c$. First what is the blocking matrix?

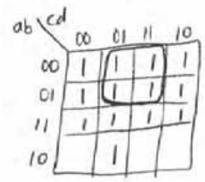
$B = 1011$
$\quad 1001$
$\quad 1010$

Let's see what happens when we simply choose a row in $B$ to find $c^+$

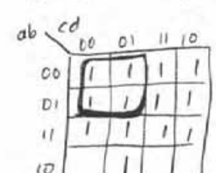
$B = 1011 \rightarrow c^+ = 0201$



$B = 1001 \rightarrow c^+ = 0221$



$B = 1010 \rightarrow c^+ = 0202$



By choosing one entry in $B$, none of the resulting $c^+$ are prime implicants (they are just implicants)
Need minimum cover of $L$ to obtain a prime implicant (i.e. $L = \{1\}$, $c^+ = 0222$)

# Covering Matrix

**Covering Matrix**

Assuming a single-output function, C is a 0-1 matrix determined by c (cube to be expanded) and F (on-set). Rows are in one-to-one correspondence with the cubes of $F^i$. The elements of $\mathbb{C}$ are defined as follows

$$\mathbb{C}_{ij} = \begin{cases} 1 & \text{if } c_j = 1 \text{ and } F_{ij} \neq 1, \text{ or} \\ & c_j = 0 \text{ and } F_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

* note covering matrix denoted by double C ($\mathbb{C}$) or capital C

EXAMPLE 6

F = 101
020
002
100

c = 002

$c_3 = 2 / F_{1,3} = 1 \therefore \mathbb{C} = 0$

$$\mathbb{C} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$c_2 = 0 / F_{2,2} = 2 \therefore c = 1$

$c_2 = 0 / F_{4,2} = 0 \therefore c = 0$

$c_1 = 0 / F_{4,1} = 1 \therefore c = 1$



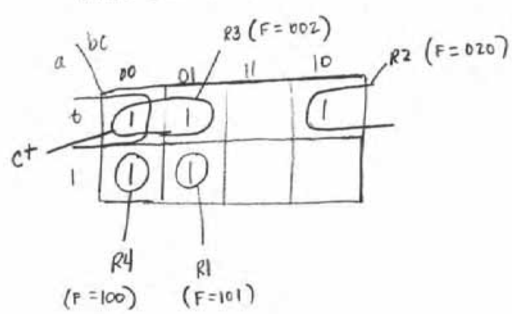
what does the covering matrix do?

A "1" in the covering matrix means if column j is in the minimum column cover (L) then cube i is not covered by $c^+$

$L = \{1, 2\}$

$\mathbb{C} =$ 1 0 0 ← R1 not covered by $c^+$
0 1 0 ← R2 not covered by $c^+$
0 0 0 ✓ R3 covered by $c^+$
1 0 0 ← R4 not covered by $c^+$

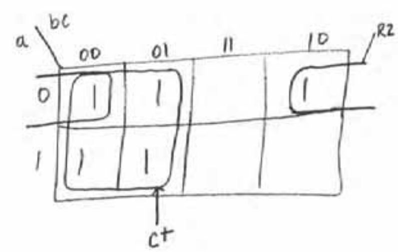
these columns in L




$L = \{2\}$

$\mathbb{C} =$ 1 0 0 ✓ R1 covered by $c^+$
0 1 0 ← R2 not covered by $c^+$
0 0 0 ✓ R3 covered by $c^+$
1 0 0 ✓ R4 covered by $c^+$

this column in L



OBJECTIVE Select minimum set of columns to add to L containing every row of B but as few rows of $\mathbb{C}$

ensures resulting cube $c^+$ is prime

ensures as many cubes as possible are included in expanded cube

$$B = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \mathbb{C} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$L = \{2\}$ covers every row of B (i.e valid column cover)

Cubes 1, 3, 4 are covered by $c^+$



$\boxed{\text{Reducing Blocking Matrix and Covering Matrix}}$

Lowering set L built by adding one column at a time. When a new column is added B and $\mathbb{C}$ can be reduced to ensure efficient processing

column j added to L (ELIM1 procedure)

↳ column deleted from B / all rows with a "1" in column deleted

↳ column deleted from $\mathbb{C}$ / all rows with a "1" in column deleted

$$B = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad B = \begin{matrix} \cancel{1\ 1\ 0} \\ \cancel{0\ 1\ 0} \end{matrix} \qquad B = \phi$$

$$\mathbb{C} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \quad \text{Add 2 to L} \quad \mathbb{C} = \begin{matrix} 1 & \cancel{0} & 0 \\ \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & \cancel{0} & 0 \\ 1 & \cancel{0} & 0 \end{matrix} \quad \text{simplify} \quad C = \begin{array}{cc} 1 & 3^{*} \\ \hline 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{array}$$

* keep track of col headings as you reduce

Some columns <u>must</u> be included in L to guaruntee disjointness of $C^+$ and R.

↳ if a row in B has only one non-zero entry, say $B_{ij}$, then column j must appear in L (only way this row is covered)

$$B = \begin{matrix} 1010 \\ 1001 \\ 0101 \\ 0100 \end{matrix}$$

0100 ← row with one non-zero entry
we say column 2 is essential ($X_E$)

you'll see code in EXPAND1 that looks for this

$X_E \leftarrow$ ESSENTIAL (B) // rows with single non-zero entry are essential
$L \leftarrow L \cup X_E$ // essential columns added to lowering set
$(B, \mathbb{C}) \leftarrow$ ELIM1 // reduce B and $\mathbb{C}$ accordingly



## Raising Set (RA) and Maximum Feasible Covering (MFC) set

Elements in raising set correspond to columns of C that are "raised" to 2, and excluded from the lowering set

Choose RA as large as possible to get largest cover (while still valid)

EXAMPLE 7

$C = 101$ $L = \{1\}$ $C^+ = 1\ 2\ 2$
$RA = \{2, 3\}$

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## When adding columns to RA (ELIM2 procedure)

- ↳ Remove columns from B and $\mathbb{C}$
- ↳ Any row in $\mathbb{C}$ that becomes all zeros are removed (cube covered)

## We may also have columns in B become all 0 (INESSENTIAL PROCEDURE)

- ↳ Column is inessential
- ↳ Remove cols of all 0's in B and add to RA

### EXAMPLE 8

$$B = \begin{array}{c} 1\,2\,3 \\ 110 \\ 010 \end{array} \qquad C = \begin{array}{c} 1\,2\,3 \\ 100 \\ 010 \\ 000 \\ 100 \end{array}$$

Add col 1 to RA

RA = {1}

$\xrightarrow{\text{ELIM2}}$

$$B = \begin{array}{c} 1\,2\,3 \\ 110 \\ 010 \end{array} \qquad C = \begin{array}{cl} 1\,2\,3 & \\ \cancel{100} & \text{row of 0's} \\ 010 & \\ \cancel{000} & \text{row of 0's} \\ 100 & \text{row of 0's} \end{array}$$

$\longrightarrow$

$$B = \begin{array}{c} 2\,3 \\ 10 \\ 10 \end{array} \qquad C = \begin{array}{c} 2\,3 \\ 10 \end{array}$$

$\xrightarrow{\text{INESSENTIAL}}$

inessential row in B

RA = {1} + {3} = {1, 3}

$$B = \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \qquad C = \begin{array}{c} 2 \\ 1 \end{array}$$

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When do we add to RA? If no row singletons (essentials in B) we look for row singletons in $\mathbb{C}$

In EXPAND1 we'll see the following

$$J^{1*} \leftarrow \text{MFC}\ (B, \mathbb{C})$$

### EXAMPLE 9

$$B = \begin{array}{c} 1\,2\,3 \\ 110 \\ 101 \end{array} \qquad \mathbb{C} = \begin{array}{c} 1\,2\,3 \\ 101 \\ 010 \end{array}$$

No essentials in B so we add col 2 to RA to cover this cube

$$B = \begin{array}{c} 1\,2\,3 \\ 110 \\ 101 \end{array} \qquad \mathbb{C} = \begin{array}{c} 1\,2\,3 \\ 100 \\ 010 \\ 101 \end{array}$$

Two row singletons, which column do we choose?

If we choose col 1, we can cover 2 rows in $\mathbb{C}$

If we choose col 2, we can cover 1 row in $\mathbb{C}$

RA = {1}

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What happens when no row singletons in $\mathbb{C}$?

$$J'^{*} \leftarrow MFC(B, \mathbb{C})$$
$$\text{if } (|J'^{*}| = 0) \ J'^{*} \leftarrow EG(\mathbb{C})$$

We call the End game procedure

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# End Game

Situation in which no more cubes in F may be covered in the expansion of c, but some columns remain to be chosen. There are three such scenarios:

**B≠Φ and C≠Φ (both not empty)**

- $c^+$ can't cover any remaining cubes no matter how L is selected, so instead we try to increase the likelihood that the partial cover will help another expanded cube and lead to irredundant cover elimination later on
- ACTION – Choose a single column in C with the maximal number of 1's and add to RA

**B≠Φ and C=Φ (C is empty)**

- Covering matrix is empty so all cubes have been covered
- ACTION – finish covering B, minimum cover algorithm called to find L, then terminate

**B = Φ (B is empty)**

- Blocking matrix is empty so nothing will interfere with off set, no reason to lower any of the remaining terms
- ACTION – Add remaining columns to RA, then terminate

put it all together and we get expand1 subroutine

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# Expand1

```
EXPAND1(c, F, R)
begin
   (B, C) ← MATRICIES(c, F, R)                      // construct blocking and covering matricies
      L ← Φ; RA ← Φ                                 // initialize lowering set (L) and raising set (RA)

   N ← NCOLS(B)                                     // find number of columns in B
   while (|L|+|RA|< N and B≠Φ and C≠Φ) begin       // process all columns (variables) to be in L or R
      X_E ← ESSENTIAL(B)                            // if column essential add to the lower set and remove from B,C
      L ← L ∪ X_E
      (B, C) ← ELIM1(B, C, X_E)

      J^i* ← MFC(B, C)                              // find columns to add to raising set while ensuring rows of B
      if( |J^i*| = 0 ) J^i* ← EG(C)                 // are covered (maximum feasible covering set)

      X_I ← INESSENTIAL(B)                          // if inessential columns generated add to the raising set
      RA ← RA ∪ J^i* ∪ X_I

      (B,C) ← ELIM2(B, C, J^i* ∪ X_I)               // remove columns added to raising set from B, C and any
   end                                              // zero row found in C

   if( B≠Φ ) then L← MINLOW(B)                      // special case, B is not empty but don't meet conditions above
   W ← {F^i ∈ F | C_ij = 0 for all j ∈ L}
   return (W, c+(L,c))                              // return prime implicant generated (c+) and the cube it covers (W)
end
```

Let's try an example



EXAMPLE 10 Expand F

$$F = \begin{matrix} 010 \\ 101 \\ 011 \\ 120 \end{matrix} \qquad R = \begin{matrix} 111 \\ 002 \end{matrix}$$

process cubes in decreasing order (largest cubes first -- most 2's)

$c_1 = 120$
$c_2 = 010$
$c_3 = 101$
$c_4 = 011$

Iteration 1 $C = 120$

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$

↑ ↑ singleton rows col essential

$L = \{1, 3\}$
Both rows of B have single 1's so column 1 and 3 are essential and added to L

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ \cancel{0} & 0 & \cancel{1} \\ \cancel{1} & 0 & \cancel{0} \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ \cancel{1} & 0 & \cancel{0} \\ \cancel{0} & 0 & \cancel{1} \\ \cancel{1} & 0 & \cancel{1} \\ \cancel{0} & 0 & \cancel{0} \end{array}$$

Reduce B and $\mathbb{C}$
- Remove col from B / remove any row with 1 in removed col
- Remove col from $\mathbb{C}$ / remove any row with 1 in removed col

$B = \phi \qquad \mathbb{C} = \overset{2}{0}$

NO MFC so we use end game
$B = \phi$, Remaining cols added to RA
$RA = \{2\}$

All columns assigned to L or RA

$C = 120 \qquad L = \{1, 3\}, \; RA = \{2\} \longrightarrow C^+ = 120$

Add $C^+$ to F, remove any cube covered by $C^+$ (110 or 100)



$$F = \begin{matrix} 010 \\ 101 \\ 011 \\ 120 \end{matrix} \qquad R = \begin{matrix} 111 \\ 002 \end{matrix}$$

(no change)

Iteration 2: $C = 010$

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

↑ singleton row col essential

$L = \{2\}$
Last row of B has a single 1 so column 2 is essential and added to L

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ 1 & \cancel{0} & 1 \\ \cancel{0} & \cancel{1} & \cancel{0} \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & \cancel{0} & 0 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ 0 & \cancel{0} & 1 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array}$$

Reduce B and $\mathbb{C}$
- Remove col from B / remove any row with a 1 in removed col
- Remove col from $\mathbb{C}$ / remove any row with a 1 in removed col

$$B = \begin{array}{cc} 1 & 3 \\ 1 & 1 \end{array} \qquad \mathbb{C} = \begin{array}{cc} 1 & 3 \\ 0 & 0 \\ 0 & 1 \end{array}$$

MFC = 3
only way to include term in $\mathbb{C}$ is to include column 3 in RA
IF col 3 added to RA we can still cover B with remaining col
$RA = \{3\}$

$$B = \begin{array}{cc} 1 & 3 \\ 1 & \cancel{1} \end{array} \qquad \mathbb{C} = \begin{array}{cc} 1 & 3 \\ 0 & \cancel{0} \\ 0 & \cancel{1} \end{array}$$

Reduce B and C
- Remove col from B and $\mathbb{C}$
- Remove any row of all 0's in $\mathbb{C}$

$$B = \begin{array}{c} 1 \\ 1 \end{array} \qquad \mathbb{C} = \begin{array}{c} 1 \\ 0 \\ \cancel{0} \end{array}$$

$B = \overset{1}{1} \qquad \mathbb{C} = \phi$

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$B = \begin{array}{c} 1 \\ 1 \end{array}$ $\mathbb{C} = \phi$

↑ singleton row

$L = L + \{1\} = \{1,2\}$

B has a singleton row, its essential

$B = 1$ $\mathbb{C} = \phi$

Reduce B and $\mathbb{C}$
- Remove col from B/remove any row with a 1 in removed col

$B = \phi$ $\mathbb{C} = \phi$

All columns added to L or RA

$c = 010$ $L = \{1,2\}$, $RA = \{3\}$ $\longrightarrow$ $c^+ = 012$

Add $c^+$ to F, remove cube covered by $c^+$ (010, 011)

$F = \cancel{010}$, 101, $\cancel{011}$, 120, 012

$F = 101$, 120, 012

$R = 111$, 002

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Iteration 3: $c = 101$

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

↑↑ singleton rows, cols essential

$L = \{1,2\}$

Essential columns in B added to L

$$B = \begin{array}{ccc} 1 & 2 & 3 \\ \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{0} \end{array} \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ \cancel{0} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{array}$$

Simplify B and $\mathbb{C}$
- remove cols from B and $\mathbb{C}$/ remove rows with 1 in removed co

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$B = \phi$ $c = \overset{3}{0}$

NO MFC, so we look to end game
$B = \phi$, remaining cols added to RA
$RA = \{3\}$

$B = \phi$ $c = \overset{3}{\cancel{0}}$

Simplify B and $\mathbb{C}$
- Remove cols in B and $\mathbb{C}$
- Remove rows of all 0 in B

$B = \phi$ $c = \phi$

All columns added to L or RA

$c = 101$ $L = \{1,2\}$, $RA = \{3\}$ $\longrightarrow$ $c^+ = 102$

$c^+$ added to F, remove any cubes covered by $c^+$ (100, 101)

$F = \cancel{101}$, 120, 012, 102

$F = 120$, 012, 102

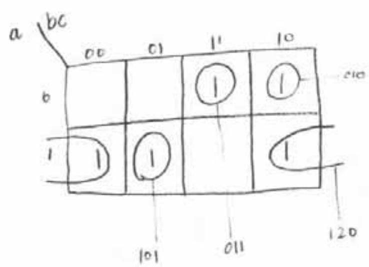
$R = 111$, 002

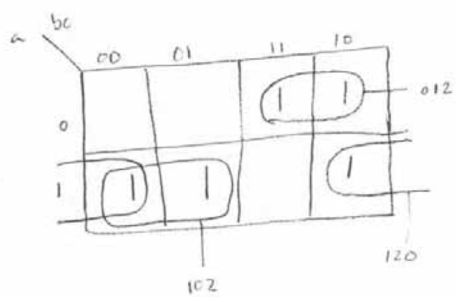
Done! No need for iteration 4, $c_4 = 011$ was removed so all cubes processed!

How did we do?

original $F = \begin{matrix} 010 \\ 101 \\ 011 \\ 120 \end{matrix}$

after expand $F = \begin{matrix} 120 \\ 012 \\ 102 \end{matrix}$






## EXAMPLE 11 Expand F

$F = \{ \bar{x}_1\bar{x}_2x_4,\ \bar{x}_1x_2x_4,\ x_1\bar{x}_2\bar{x}_3,\ x_1\bar{x}_2\bar{x}_4 \}$

$R = \{ x_1x_2,\ x_2\bar{x}_4 \}$

$F = \begin{matrix} 0021 \\ 0121 \\ 1002 \\ 1020 \end{matrix}$ $R = \begin{matrix} 1122 \\ 2120 \end{matrix}$

all cubes same size, processed in the following order

$c_1 = 0021$
$c_2 = 0121$
$c_3 = 1002$
$c_4 = 1020$

iteration 1: $c = 0021$

$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}$ $\mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$

No singleton rows in B

Second row in $\mathbb{C}$ is singleton

$RA = \{2\}$

col 3 in B is inessential

$RA = \{2\} + \{3\} = \{2,3\}$

$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}$ $\mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$

Reduce B and $\mathbb{C}$

- Remove cols from B and $\mathbb{C}$

$B = \begin{array}{cc} 1 & 4 \\ 1 & 0 \\ 0 & 1 \end{array}$ $\mathbb{C} = \begin{array}{cc} 1 & 4 \\ \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} \\ 1 & 1 \\ 1 & 1 \end{array}$

- Remove rows of all 0s from $\mathbb{C}$

$$B = \begin{array}{cc} 1 & 4 \\ 1 & 0 \\ 0 & 1 \end{array} \qquad \mathbb{C} = \begin{array}{cc} 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{array}$$

↑ ↑ singleton rows

singleton rows in B

$L = \{1,4\}$

$$B = \begin{array}{cc} 1 & 4 \\ 1 & 0 \\ \cancel{0} & \cancel{1} \end{array} \qquad \mathbb{C} = \begin{array}{cc} 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{array}$$

Reduce B and $\mathbb{C}$
- remove cols in B and $\mathbb{C}$/ remove rows with 1 in removed col

$B = \phi \qquad \mathbb{C} = \phi$

All columns added to L or RA

$c = 0021 \qquad L = \{1,4\} \longrightarrow c^+ = 0221$

$RA = \{2,3\}$

Add $c^+$ to F, remove any cubes covered by $c^+$

| F = | F = | R = |
|---|---|---|
| ~~0021~~ | 1002 | 1122 |
| ~~0121~~ | 1020 | 2120 |
| 1002 | 0221 | |
| 1020 | | |
| 0221 | | |



iteration 2 · $c = 1002$ ($C_2 = 0121$ was removed - why expand a cube that's not there)

$$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}$$

↑ singleton row

Singleton row in B

$L = \{2\}$

$$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \end{array} \qquad \mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{1} & \cancel{0} \end{array}$$

Reduce B and $\mathbb{C}$
- remove cols in Band $\mathbb{C}$/ remove rows with a 1 in removed col

$$B = \phi \qquad C = \begin{array}{ccc} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$$

MFC = 3

$RA = \{3\}$

$$B = \phi \qquad C = \begin{array}{ccc} 1 & & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} \end{array}$$

row of 0s

row of 0's after elim 3

Reduce Band $\mathbb{C}$
- remove cols in B and $\mathbb{C}$
- remove rows of 0's in $\mathbb{C}$

$B = \phi \qquad C = \phi$

Wait! Col 1 and 4 havent been assigned.

what should we do?

If B or $\mathbb{C} = \phi$ fall out of loop

$c = 1002 \qquad L = \{2\} \longrightarrow c^+ = 2022$

$RA = \{3\}$

* expansion based on L

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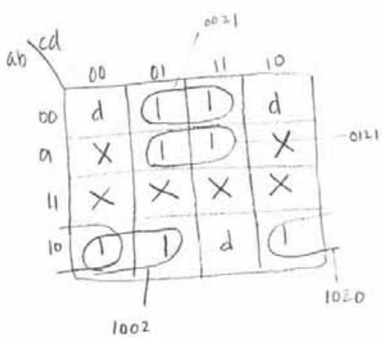
Add $c^+$ to F, remove cubes covered by $c^+$

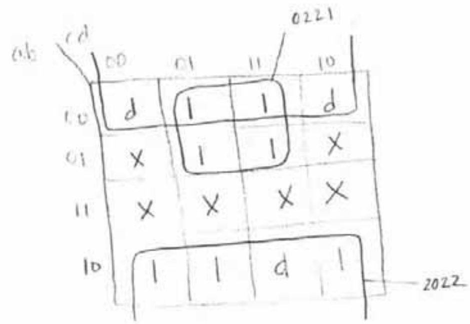
F= ~~1002~~ F = 0221 R = 1122
~~1020~~ 2022 2120
0221
2022

Done! All cubes expanded (or removed)

How did we do?

original F = 0021 R=1122 expanded F = 0221
0121 2120 2022
1002
1020




(30)

EXAMPLE 12 expand F

F= 000 R = 201
100 121
210

cubes processed in following order
$C_1$ = 210
$C_2$ = 000
$C_3$ = 100

iteration 1: c = 210

B = 011
001
singleton row

$\mathbb{C}$ = 010
010
000

Singleton row in B
L = {3}

B = ~~011~~
~~001~~

$\mathbb{C}$ = 010
010
000

Reduce B and $\mathbb{C}$
- remove cols in B and $\mathbb{C}$
- remove rows with 1 in removed cols

B = $\phi$

$\mathbb{C}$ = 01
01
00

MFC = 2
RA = {2}

B = $\phi$

C = 01
01
00

Reduce B and $\mathbb{C}$
- remove cols in B and $\mathbb{C}$
- remove rows of 0s from $\mathbb{C}$

B = $\phi$

C = ~~0~~
~~0~~
~~0~~
zero rows

B ≠ 0 C = $\phi$

Wait! col 1 not assigned
doesnt matter, fall out of loop

$c = 210 \qquad L = \{3\} \longrightarrow c^{+} = 220$

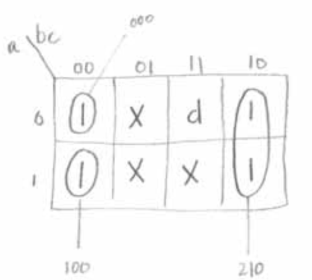
$RA = \{2\}$

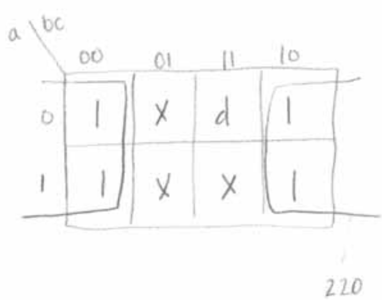
add $c^{+}$ to F, remove cubes covered by $c^{+}$

$F = \begin{array}{l} \cancel{000} \\ \cancel{100} \\ \cancel{210} \\ 220 \end{array} \qquad F = 220 \qquad R = \begin{array}{l} 201 \\ 121 \end{array}$

How did we do?

original $F = \begin{array}{l} 000 \\ 100 \\ 210 \end{array} \quad R = \begin{array}{l} 201 \\ 121 \end{array}$ after expand $F = 220$






EXAMPLE 13 expand F

$F = \begin{array}{l} 0022 \\ 1212 \\ 0122 \end{array} \qquad R = 1202$

cubes processed in the following order

$c_1 = 0022$

$c_2 = 1212$

$c_3 = 0122$

iteration 1: $c = 0022$

$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$

row singleton

Singleton row in B

$L = \{1\}$

$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \end{array} \qquad \mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \cancel{0} & 0 & 0 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & 1 & 0 & 0 \end{array}$

Reduce B and $\mathbb{C}$

$B = \phi \qquad \mathbb{C} = \begin{array}{ccc} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$

MFC = 2

$RA = \{2\}$

$B = \phi \qquad \mathbb{C} = \begin{array}{ccc} 2 & 3 & 4 \\ \cancel{0} & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{0} \end{array}$

Reduce B and $\mathbb{C}$

$B = \phi \qquad \mathbb{C} = \phi$

B or $\mathbb{C}$ becomes $\phi$

fall out of loop

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$c = 0022$ $\quad L = \{1\}$ $\longrightarrow$ $c^+ = 0222$

$RA = \{2\}$

add $c^+$ to F, remove any cubes covered by $c^+$

F = ~~0022~~
1212
~~0122~~
0222

F = 1212
0222

R = 1202

iteration 2 $\quad c = 1212$

$$B = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{array} \qquad \mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

singleton row

Singleton row in B
L = {3}

$$B = \cancel{0010} \qquad \mathbb{C} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} \end{array}$$

Reduce B and $\mathbb{C}$

$$B = \phi \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}$$

NO MFC, call end game
B = $\phi$ so remaining cols go to RA
RA = {1, 2, 4}

$$B = \phi \qquad \mathbb{C} = \begin{array}{ccc} 1 & 2 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}$$

Reduce B and $\mathbb{C}$

$B = \phi \qquad C = \phi$

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$c = 1212$ $\quad L = \{3\}$ $\longrightarrow$ $c^+ = 2212$

$RA = \{1, 2, 4\}$

add $c^+$ to F, remove and cubes covered by $c^+$

F = ~~1212~~
0222
2212

F = 0222
2212

R = 1202

How did we do?

original F = 0022
1212
0122

R = 1202

after expand F = 0222
2212