# Lecture 10 D

# Espresso - Complement

(1)

Not all functions are unate. How do we complement non-unate functions? Shannon Expansion

$$F(a,b,c) = aF_a + a'F_{a'}$$
$$= abF_{ab} + ab'F_{ab'} + a'bF_{a'b} + a'b'F_{a'b'}$$

we can keep generating cofactors until we find a unate function then complement using unate complement and merge results

② 

How do we determine a cofactor in compat cubical form?

Example 1 Given a set of cubes G and cube p, determine $G_p$ the cofactor of G with respect to (w.r.t) p

$$G = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad p = \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix} \qquad G_p = \begin{bmatrix} 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

How did we do that?

cubes with "11" in the first two columns go into the cofactor

1102 // ABC' is in $G_p$

0120 // A'BC' is NOT in $G_p$

1111 // ABCD is in $G_p$

also cubes with "22" in the first two columns go into the cofactor

if we had 2210 // CD' this would also go into $G_p$

In the cofactor we remove (or factor out) the cube the cofactor is with respect to, and leave remaining literals

ABC'
ABCD $\longrightarrow$ $G_p$ = C' + CD

**02 $\rightarrow$ 2202
**11 $\rightarrow$ 2211

③

so how do we formalize the cofactor?

Given a set of cubes G = {$c^1$, ..., $c^L$} and a cube p, all with n entries in the input part and m entries in the output part, the cofactor of G with respect to p, $G_p$, is a set of cubes (possibly empty) obtained by computing the cofactor of each of the cubes in the cover G.

The cofactor of $c^i$ with respect to p as the cube with components

$$(c^i_p)_k = \begin{cases} \Phi & \text{if } c^i \cap p = \Phi \\ 2 & \text{if } p_k = 0 \text{ or } 1 \\ 4 & \text{if } p_k = 3 \\ c^i_k & \text{otherwise} \end{cases}$$

// if a 1/0 combination, it's empty

// $p_k$ variable is what we are factoring out

// ignore, we are dealing with single output functions

// copy whatever was in $c^i_k$

EXAMPLE 2 using shannon expansion thm, expand F wrt D

④

$F(a,b,c,d) = abc' + a'bd' + abcd$

① Let's get F into compact cubical form

$$F = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

② Let's find cofactor w.r.t D ($F_D$)

Ⓐ what are we factoring out?

$p = [2221]$ "D

Ⓑ which cubes go into cofactor?

if $c^i \cap p = \emptyset$ these cubes left out

$[1102] \cap [2221] = 1102$

$[0120] \cap [2221] = 012\emptyset$ X

$[1111] \cap [2221] = 1111$

Ⓒ of the cubes moved over, how are they transformed?

$[1102] \longrightarrow [1102]$

$[1111] \longrightarrow [1112]$

if col in p has "1/0" cofactor col has "2"

if col in p has "2" cofactor keeps existing value

$$\boxed{F_D = \begin{matrix} 1102 \\ 1112 \end{matrix}}$$

EXAMPLE 2 CONT'

⑤

③ Let's find cofactor w.r.t. D' ($F_{D'}$)

Ⓐ what are we factoring out?

$p = [2220]$ "D'

Ⓑ which cubes go into cofactor?

$[1102] \cap [2220] = 1100$

$[0120] \cap [2220] = 0120$

$[1111] \cap [2220] = 111\emptyset$ X

Ⓒ how are these cubes transformed?

$[1102] \longrightarrow [1102]$

$[0120] \longrightarrow [0122]$

colums where p = 1/0 are now "2"

colums where p = 2 are left alone

$$\boxed{F_{D'} = \begin{matrix} 1102 \\ 0122 \end{matrix}}$$

④ Let's put it all together

$$F = DF_D + D'F_{D'}$$

$$F = [2221]\begin{bmatrix} 1102 \\ 1112 \end{bmatrix} + [2220]\begin{bmatrix} 1102 \\ 0122 \end{bmatrix}$$

## How can I check my solution?

① Boolean Algebra - translate cube back to Boolean logic equation and double check

$$F = [2221]\begin{bmatrix}1102\\1112\end{bmatrix} + [2220]\begin{bmatrix}1102\\0122\end{bmatrix}$$

$$\downarrow$$

$$F = (d)(abc' + abc) + (d')(abc' + a'b)$$

$$F = abc' + a'bd' + abcd$$

$$\downarrow$$

$$F = (d)(abc' + abc) + (d')(abc' + a'b)$$

equations match ✓

② You can multiply out the cubes to make sure we end up with something equivalent to F.

$$[2221]\cdot\begin{bmatrix}1102\\1112\end{bmatrix} = \begin{bmatrix}1101\\1111\end{bmatrix}$$

$$[2220]\cdot\begin{bmatrix}1102\\0122\end{bmatrix} = \begin{bmatrix}1100\\0120\end{bmatrix}$$

$$\begin{bmatrix}1101\\1111\end{bmatrix} + \begin{bmatrix}1100\\0120\end{bmatrix} = \begin{bmatrix}1101\\1111\\1100\\0120\end{bmatrix}$$

$$\begin{bmatrix}1101\\1111\\1100\\0120\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}1102\\0120\\1111\end{bmatrix}$$

Yes!

1101 and 1100 can be combined to 1102, giving us the original cover



## EXAMPLE 3: Find $F_B$, $F_A$, $F_{C'}$

$F(a,b,c) = ab' + bc + ac'$

$$F = \begin{matrix}1&0&2\\2&1&1\\1&2&0\end{matrix}$$

**Find $F_A$**

$p = [122]$

check if cube included

$[102] \cap [122] = 102$

$[211] \cap [122] = 111$

$[120] \cap [122] = 120$

translate

$[102] \rightarrow [202]$

$[211] \rightarrow [211]$

$[120] \rightarrow [220]$

$$\boxed{\begin{matrix}F_A = 202 & // b' \\ 211 & // bc \\ 220 & // c'\end{matrix}}$$

**Find $F_B$**

$p = [212]$

check if included

$[102] \cap [212] = 1\phi2$ X

$[211] \cap [212] = 211$

$[120] \cap [212] = 110$

translate

$[211] \rightarrow 221$

$[120] \rightarrow 120$

$$\boxed{\begin{matrix}F_B = 221 & // C \\ 120 & // AC'\end{matrix}}$$

**Find $F_{C'}$**

$p = [220]$

check if included

$[102] \cap [220] = 100$

$[211] \cap [220] = 21\phi$ X

$[120] \cap [220] = 120$

translate

$[102] \rightarrow 102$

$[120] \rightarrow 122$

$$\boxed{\begin{matrix}F_{C'} = 102 & // AB' \\ 122 & // A\end{matrix}}$$

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To complement Espresso has fast/efficent complementor for unate functions

How to go from non-unate to unate? Shannon Expansion

shannon expansion is repeated until cofactors become unate

in the case of multiple output functions, espresso will extract single output functions, complement these functions, and put them back together

```
COMPLEMENT(F, D)
begin
  R ← Φ

  for (i=1, ..., m)                // for each output extract single output function
  begin
    (F_i, D_i) ← EXTRACT(F, D, i)
    R_i ← COMP1(F_i ∪ D_i)         // find complement of single output function
    R ← R, R_i                     // merge results
  end

  Reutrn (R)
End
```

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COMP1 is a single output complementor

tries to take advantage of several special conditions

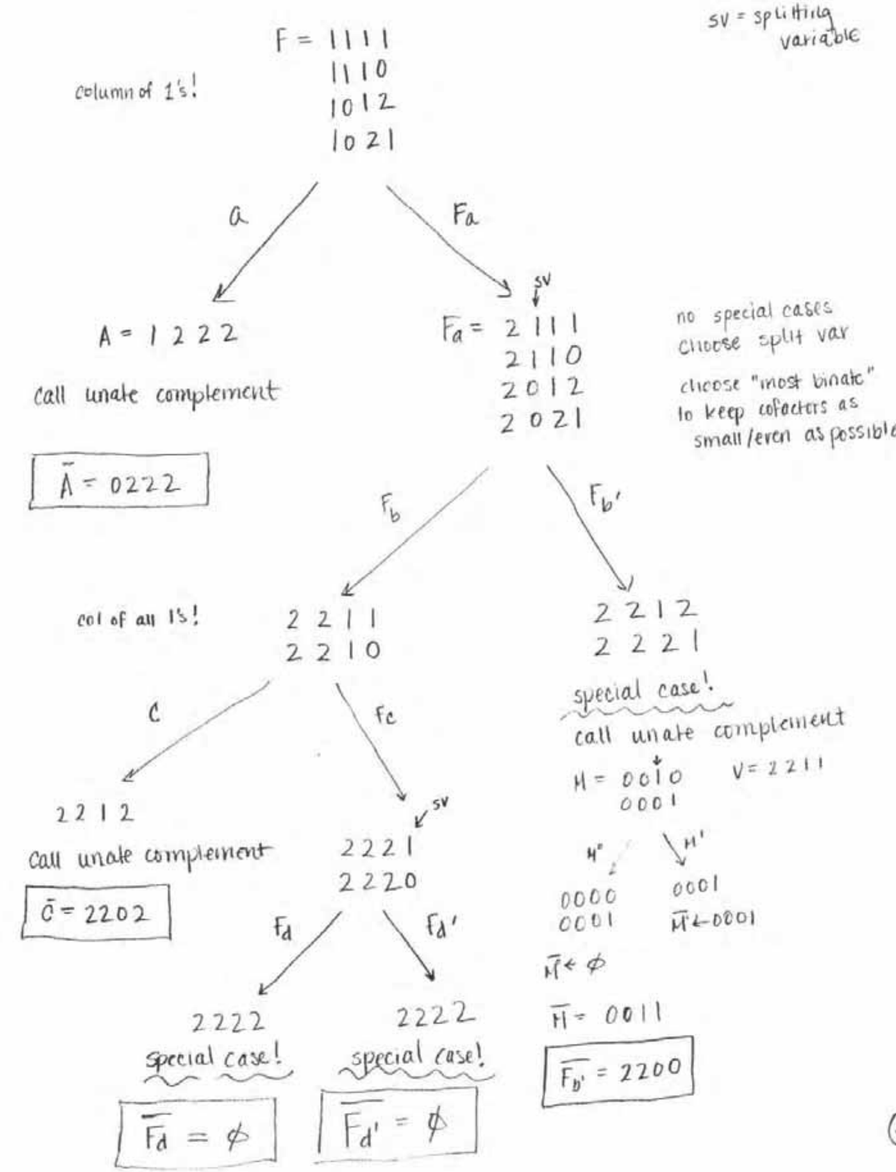
otherwise generic decomposition is performed

```
COMP1(F)
begin
  if (row of all 2's) Return (R ← Φ)                      // special case – contain universe, complement is empty
  if (F unate) Return (R ← UNATE_COMPLEMENT(F))           // special case – unate, call unate complement
  c ← F^1                                                 // special case – column of all 0/1's, extract variable c
  for(j=1, ..., n)                                        // for loop determines F_c
    begin
      for(i=2, ..., |F|)
        if(c_j ≠ F^i_j) then c_j ← 2                      // basically look at each cube and
    end
  R ← UNATE_COMPLEMENT({c})                               // variable c unate, determine complement of c
  F ← F_c                                                 // get remaining function F_c


  j ← BINATE_SELECT(F)                                    // choose splitting variable
  R ← R, MERGE_WITH_CONTAINMENT(COMP1(F_xj), COMP1(F_x̄j)) // call recursively on cofactors
  Return
end
```

## EXAMPLE 4 compute $F'$

sv = splitting variable





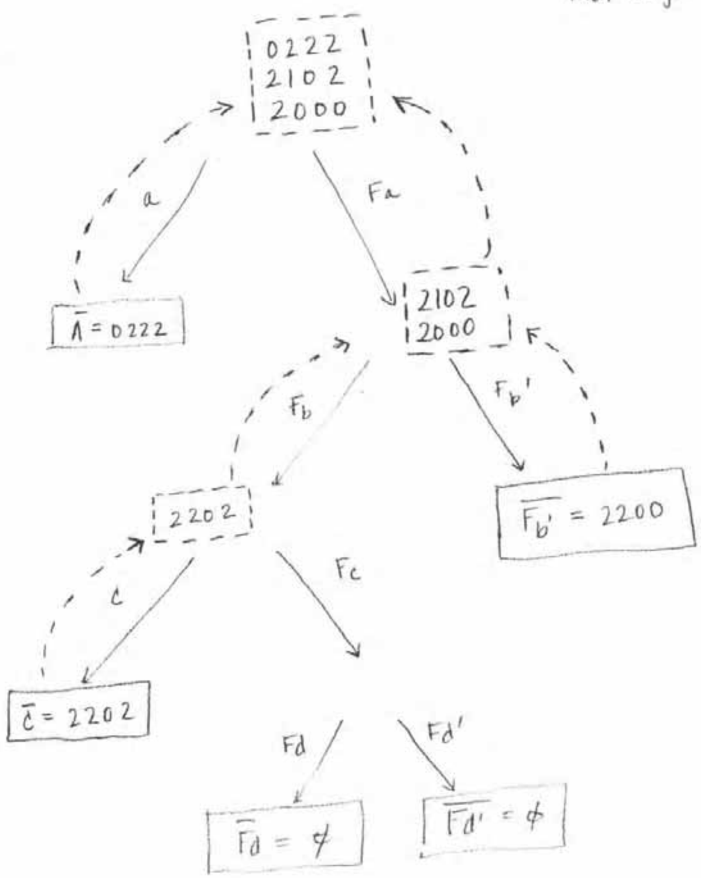

## EXAMPLE 4 CONT

now that we have found complement of all leaf nodes how do we put it back together?

2 case $\quad F = aF_a \rightarrow F' = a' + \overline{F_a}$ this is just a union of leaf nodes

$F = aF_a + a'F_{a'} \rightarrow a\overline{F_a} + a'\overline{F_{a'}}$ this adds variable back into cofactors then merges leaf nodes

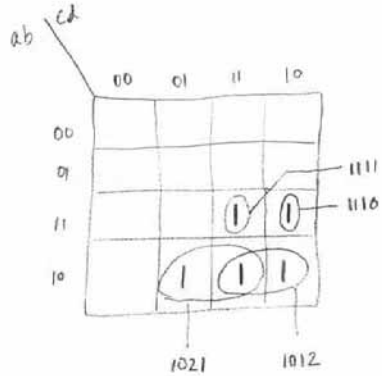
$$\boxed{\begin{array}{l} \text{SOLN:} \\ \overline{F} = 0222 \\ \quad\;\; 2102 \\ \quad\;\; 2000 \end{array}}$$

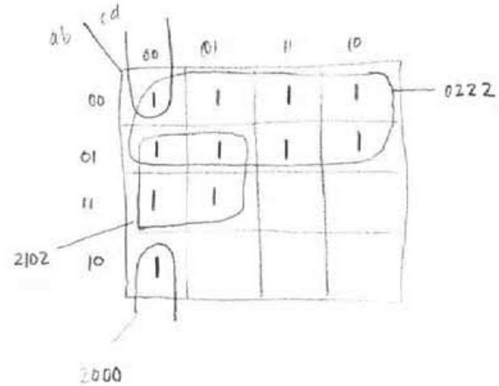


Let's check our solution

F = 1111
1110
1012
1021

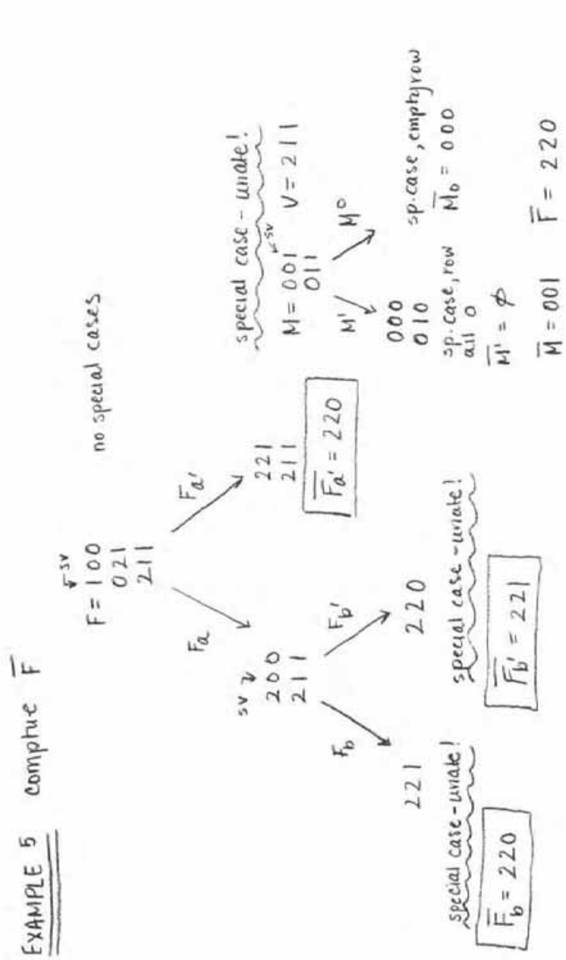
F' = 0222
2102
2000






EXAMPLE 5 comptue $\overline{F}$




special case - unate!

M = 001 (sv) V = 211
011

$M' \swarrow \quad \searrow M^0$

000
010

sp. case, row all 0

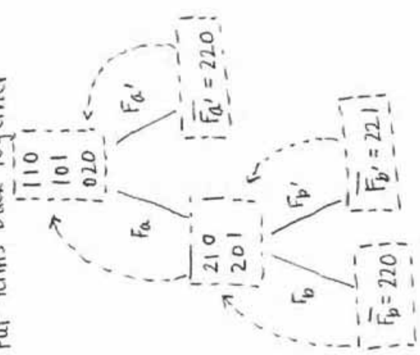
$\overline{M'} = \phi$

sp. case, empty row

$\overline{M_0} = 000$

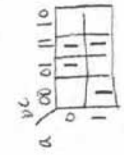
$\overline{M} = 001 \quad \overline{F} = 220$

Put terms back together

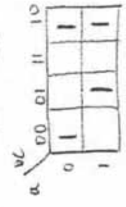



Solution
F = 110
101
020

we can also double check with k-map




F




F'

## EXAMPLE 7   Compute $\overline{F}$

$F = \begin{matrix} 1002 \\ 2011 \\ 1020 \end{matrix}$   special case - column of all 0s

$F \xrightarrow{b'} b' = 2022$   $\boxed{b'' = 2122}$

$F \xrightarrow{F_{b'}} \begin{matrix} 1202 \\ 2211 \\ 1220 \end{matrix}$ ($\downarrow$ sv)

$\xrightarrow{F_c} \begin{matrix} 2221 \\ 1220 \end{matrix}$ ($\downarrow$ sv)   $\xrightarrow{F_{c'}} \begin{matrix} 1222 \\ 1220 \end{matrix}$   $\boxed{\overline{F_{c'}} = 0222}$

$\xrightarrow{F_d} 2222$   special case   $\boxed{\overline{F_d} = \phi}$

$\xrightarrow{F_{d'}} 1222$   special case, unate   $\boxed{\overline{F_{d'}} = 0222}$

special case - unate

$M = \begin{matrix} 1000 \\ 1001 \end{matrix}$   $V = 122$

$M \xrightarrow{M'} \begin{matrix} 0000 \\ 0001 \end{matrix}$   $\overline{M'} = \phi$

$M \xrightarrow{M^0}$   $\overline{M^0} \leftarrow 000($

$\overline{M} = 1000$   $\overline{F} = 0222$

put terms back together

$\overline{F_d}\ \phi \xrightarrow{F_d}$ $0220$ $\xleftarrow{F_{d'}} \overline{F_{d'}} = 0222$

$0220 \xrightarrow{F_c} \begin{matrix} 0210 \\ 0202 \end{matrix} \xleftarrow{F_{c'}} \overline{F_{c'}} = 0222$

$b = 2122 \xrightarrow{b'} \begin{matrix} 2122 \\ 0210 \\ 0202 \end{matrix} \xleftarrow{F_{b'}} \begin{matrix} 0210 \\ 0202 \end{matrix}$

**SOLUTION**

$F' = \begin{matrix} 2122 \\ 0210 \\ 0202 \end{matrix}$

check with k-map

| ab \ cd | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | | 1 | |
| 01 | | | | |
| 11 | | | | |
| 10 | 1 | 1 | 1 | 1 |

| ab \ cd | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | | 1 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | | | | |

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## EXAMPLE 6   Compute $\overline{F}$

$F = \begin{matrix} 2200 \\ 0212 \\ 2012 \end{matrix}$ ($\downarrow$ sv)   no special cases

$F \xrightarrow{F_c} \begin{matrix} 0222 \\ 2022 \end{matrix}$   special case - unate!   $\boxed{\overline{F_c} = 1122}$

$F \xrightarrow{F_{c'}} 2220$   special case - unate!   $\boxed{\overline{F_{c'}} = 2221}$

$M = \begin{matrix} 1000 \\ 0100 \end{matrix}$ ($\downarrow$ sv)

$M \xrightarrow{M'} \begin{matrix} 0000 \\ 0100 \end{matrix}$   $\overline{M'} = \phi$

$M \xrightarrow{M^0} 0100$   $\overline{M^0} = 0100$

$\overline{M} = 1100$   $\overline{F} = 1122$

$V = 0022$

put terms back together

$\overline{F_c} = 1122 \xrightarrow{F_c} \begin{matrix} 1112 \\ 2201 \end{matrix} \xleftarrow{F_{c'}} \overline{F_{c'}} = 2221$

**SOLUTION**

$\overline{F} = \begin{matrix} 1112 \\ 2201 \end{matrix}$

double check with K-map

| ab \ cd | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | 1 | 1 |
| 01 | 1 | | 1 | 1 |
| 11 | 1 | | | |
| 10 | 1 | | 1 | 1 |

F

| ab \ cd | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | 1 | | |
| 01 | | 1 | | |
| 11 | | 1 | 1 | 1 |
| 10 | | 1 | | |

F'

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