Graphs: Many problems can be represented using graphs. In fact, all of the data structures we have discussed (lists, stacks, queues, trees, etc.) can be represented as graphs.

Basic Definitions:

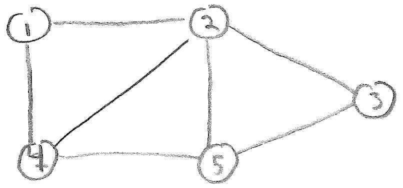
$G = (V, E)$, where V is a set of vertices (or nodes)
and E is a set of edges between vertices

An edge (u, v) specifies a connection between vertex u and vertex v.

A path is a sequence of vertices traversed by following edges between vertices. A path from a vertex u to a vertex v is a sequence of vertices $\langle v_0, v_1, \ldots, v_k \rangle$ in which $u = v_0$ and $v = v_k$ and for all $i = 1$ to k, $(v_{i-1}, v_i) \in E$.

Undirected graph contains edges that specify connections between vertices without any predecessor (source) / successor (sink) relationship. An edge (u,v) is equivalent to (v,u) in and undirected graph.

Example:



$G = (V, E)$

$V = \{1, 2, 3, 4, 5\}$

$E = \{(1,2), (1,4), (2,3), (2,4), (2,5), (3,5), (4,5)\}$

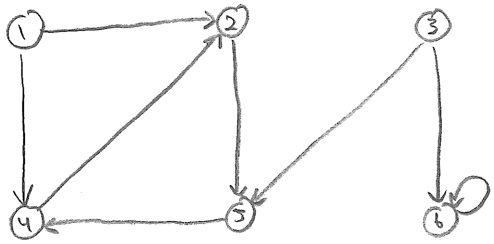
⇓ could also represent this graph as the following

$G = (V, E)$

$V = \{(2,1), (1,4), (3,2), (4,2), (2,5), (5,3), (5,4)\}$

<u>Directed Graph:</u> consists of edge that specify a directed connection from a source vertex to a sink vertex.

<u>Example:</u>



$G = (V, E)$

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{(1,2), (1,4), (2,5), (3,5), (3,6), (4,2), (5,4), (6,6)\}$

<u>Edge Weights:</u> Graphs can also be weighted in which a weight can be associated with each edge.

<u>Graph Representation:</u> Adjacency List Representation ⇒ Graph can be represented as a list of vertices where each vertex contains a list of vertices to which the vertex connects.

*Note: The following data structures for the graph are not the recommended approach for implementing a graph in C

<u>Vertex:</u>

```
struct Vertex_ {
      int id;                              } → id of node can be int, char *, etc.
      int distance;                        } → will be used in traversal and shortest path
      int color                            }     algorithms
      // we may need other data elements.    ↘ many algorithms use colors to track
                                                 status of nodes
      EdgeList edges;                      } → List of edges (see below)
      struct Vertex_ *next;                } → will be used in list of
      struct Vertex_ *prev;                      vertices (VertexList)
};
```

<u>Edge:</u>

```
struct Edge_ {
      Vertex *adjVertex;                   } → pointer to adjacent Vertex
      int weight;                          } → weight of edge
      struct Edge_ *next;                  } → will be used in list of
      struct Edge_ *prev;                        edges (EdgeList)
};
```

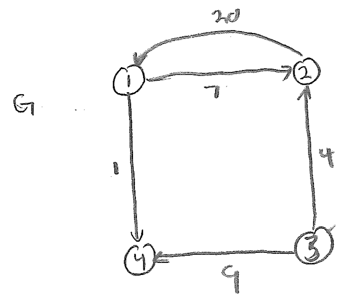
<u>Graph:</u>

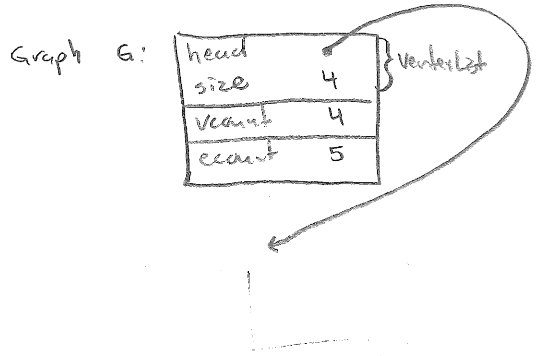
```
typedef struct Graph_ {
      VertexList vertices;
      int vcount;
      int ecount;
} Graph;
```

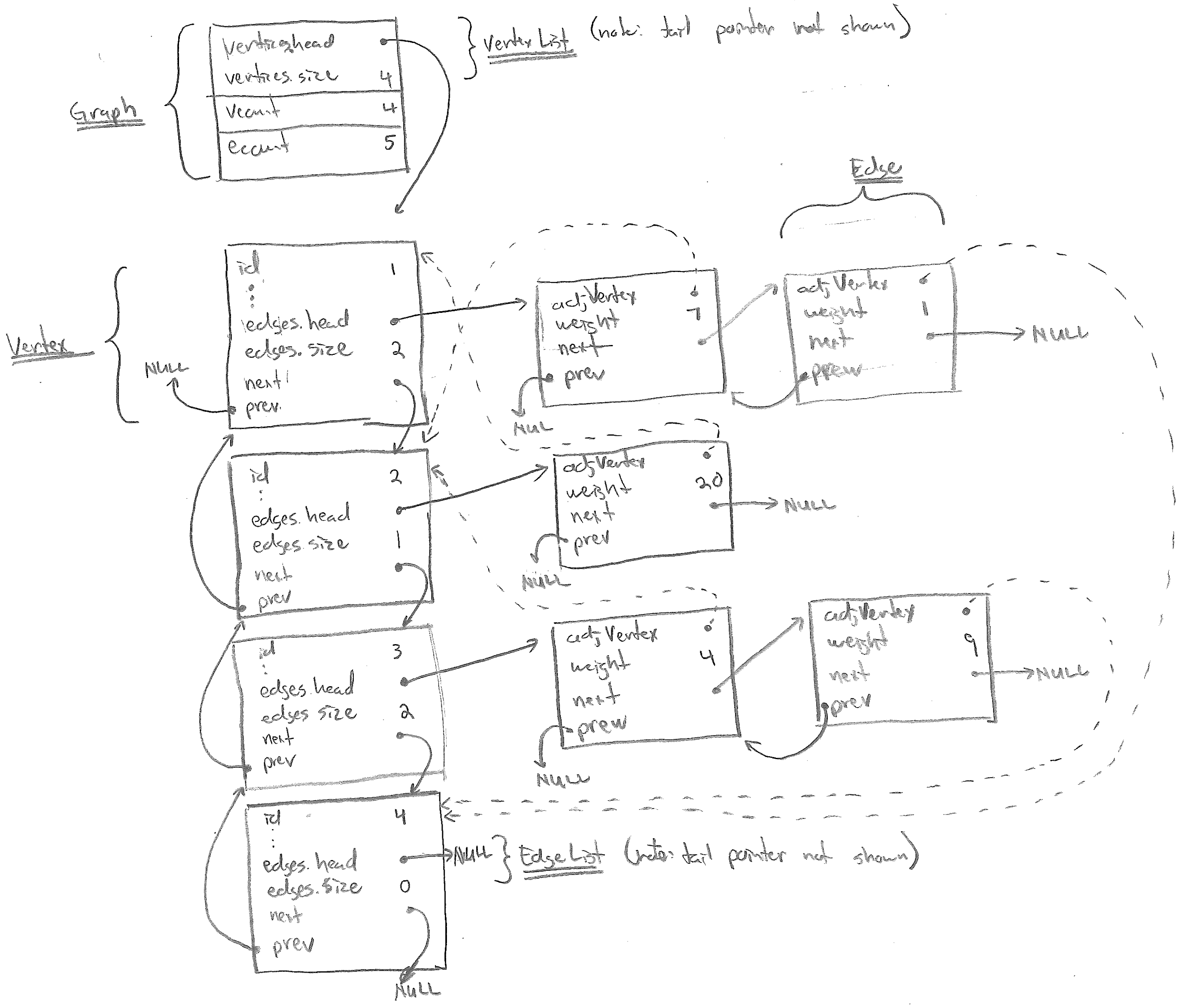
* Must include the following typedefs in shared header file (e.g. global.h) in order to avoid circular dependencies.

```
typedef struct Vertex_ Vertex;
typedef struct Edge_ Edge;
```

Example: Graph representation of directed weighted graph.







Basic Graph Traversal Algorithms:

Breadth-First Search: Given a graph G = (V,E) and a start vertex s, compute distance to all vertices v ∈ V-{s} from s

*assuming unweighted graph

```
BFS(Graph *g, Vertex *s):
  ① for each vertex u ∈ g→vertices, u != s
      a) u→color = white           ——→ white indicates a vertex has not been
                                        found or visited
      b) u→distance = ∞
  ② s→color = gray                 ——→ gray indicates a vertex has been
  ③ s→distance = 0                      found but not visited
  ④ initialize queue Q             ——→ Q is a queue holding all vertices
                                        found so far but not visited.
  ⑤ enqueue(Q, s)
  ⑥ while Q→size != 0
      a) u = dequeue(a)
      b) for each edge e ∈ u→edges
          I) v = e→adjVertex
          II) if v→color == white
              [i] v→color = gray
              [ii] v→distance = u→distance + 1
              [iii] enqueue(Q, v)
      c) u→color = black           ——→ black indicates a node has been
                                        visited
```

Depth-First Search: Given graph G = (V,E), compute discovery and finishing time for each vertex v ∈ G.

- * Does not require start vertex, but could be modified to do so
- * Assuming unweighted graph

```
DFS(Graph * g):
  ① for each vertex u ∈ g→vertices
       u→color = white
  ② time = 0
  ③ for each vertex u ∈ g→vertices
     a) if u→color == white
          I> DFS-Visit(g, u)
```

```
DFS-Visit(Graph * g, Vertex * u):
  ① u→color = gray
  ② time++
  ③ u→distance = time           ——→ discovery time
  ④ for each edge e ∈ u→edges
     a) v = e→adjVertex
     b) if v→color == white
          I> DFS-Visit(g, v)
  ⑤ u→color = black
  ⑥ time++                      ——→ finish time
  ⑦ u→finish = time
```

Note: Can be used to create a forest of depth-first trees. by maintaining predecessor point in each vertex

* Depth-first starting from a single vertex s can be implemented as:

```
  ① for each vertex u ∈ g→vertices
     a) u→color = white
     b) u→distance = ∞
  ② time = 0
  ③ DFS-Visit(g, s)
```

Shortest Path Algorithms:

- Many variants of shortest path algorithms exist (single destination, single-source, all pairs)
- Consider only single-source algorithms for now

Dijkstra's Single-Source Shortest Path: Given a graph $G = (V, E)$ and a start vertex $s \in V$, compute the shortest path from s to each vertex $v \in V$

* Assumes G is a weighted graph, such that for each edge (u,v), a weight $w(u,v)$ is specified. All edge weights must be non-negative

```
DijkstraShortestPath(Graph *g, Vertex *s):
① for each vertex v ∈ g→vertices
    a) v→distance = ∞
    b) v→pred = NULL
② s→distance = 0
③ initialize list Q
④ for all vertices v ∈ g→vertices
    a) list_insert(Q, v)
⑤ while Q→size != 0
    a) u = extract_min(Q)
    b) for each edge e ∈ u→edges
         v = e→adjVertex
         Relax(u, e)
```

→ pred is pointer to previous vertex in shortest path

→ Q is a list of vertices found so far but not visited sorted by vertices' distance. (often referred to as a priority queue or heap)

→ returns pointer to vertex in Q with smallest distance → simple implementation can use unsorted list that will be searched each time to find vertex with minimum distance

```
Relax(Vertex *u, Edge *e)
① v = e→adjVertex
② if v→distance > u→distance + e→weight
    a) v→distance = u→distance + e→weight
    b) v→pred = u
```

Runtime Complexity: $O(V^2)$ using list to implement the Q

* Can be improved to $O((V+E)\lg V)$ and even $O(V\lg V + E)$ using different data structures for the minimum priority queue.

Bellman-Ford Shortest Path: Single-source shortest path that works on graphs with negative edge weights. Can also detect if a negative-weight cycle exists

Given a graph G = (V, E) and a start vertex s ∈ V, compute the shortest path from s to each vertex v ∈ V. Returns **TRUE** if no cycles exist (i.e. graph is acyclic), **FALSE** otherwise.

```
bool
BellmanFordShortestPath (Graph *g, Vertex *s):
① for each vertex v ∈ g→vertices
      a) v→distance = ∞
      b) v→pred = NULL

② s→distance = 0

③ for i = 0 to g→vertices→size - 1            ]→ repeats process |V| times

      a) for each vertex u ∈ g→vertices       ]→ Equivalent to: for all edges (u,v) ∈ G.E
            I> for each edge e ∈ u→edges      ]
                  i) v = e→adjVertex
                  ii) Relax(u, e)

④ for each vertex u ∈ g→vertices
      a) for each edge e ∈ u→edges
            I> v = e→adjVertex
            II> if v→distance > u→distance + e→weight, return FALSE ]– condition that indicates
                                                                       a negative weight cycle
                                                                       exists.
⑤ return TRUE
```

Runtime Complexity: $O(VE)$