Pulse Energy Probability Density Functions for Long-Haul Optical Fiber Transmission Systems by Using Instantons and Edgeworth Expansion

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***Abstract*—In this work, we use a new approach to model pulse energy in long-haul optical fiber transmission systems. Existing approaches for obtaining probability density functions (pdfs) rely on numerical simulations or analytical approximations. Numerical simulations make far tails of the pdfs difficult to obtain, while analytical approximations are often inaccurate, as they neglect nonlinear interaction between pulses and noise. Our approach combines the instanton method from statistical mechanics to model far tails of the pdfs, with numerical simulations to refine the middle part of the pdfs. We combine the two methods by using an orthogonal polynomial expansion constructed specifically for this problem. We demonstrate the approach on an example of a specific submarine transmission system.**

***Index Terms*—Edgeworth expansion, instantons, optical fiber transmission, probability density function (pdf).**

I. Introduction

INVESTIGATION of error statistics in high-speed optical fiber communication systems is a fundamental task. The nonlinear nature of the propagation of light, and nonlinear intersymbol interference (ISI) between neighboring pulses coupled with noise, make this task very challenging.

Due to ISI, it is necessary to study probability density functions (pdfs) of signal samples corresponding to longer bit configurations and not just individual bits. Existing approaches for modeling these pdfs rely either on extensive numerical simulations [1], [6] or on simplified and often inaccurate analytical approximations (for a discussion see [17] and references therein). Numerical approaches approximate the middle part of distributions well, but far tails of pdfs are very hard to obtain numerically. All the existing analytical approaches neglect nonlinear interaction between pulses and noise (which is implicitly incorporated in numerical modeling). This leads to pdf approximations that are applicable only under very severe restrictions in terms of system speed, distance, and types of fiber used (see [3] and [15]).

Several recent papers [3], [4], [10], [11], [13] used Karhunen–Loève (KL) series expansion to determine pdfs. In cases where the covariance function is known, this method works well [3]. In the case of a general optical fiber communication system, covariance between received pulses is unknown and the KL expansion approach has several drawbacks. First, a covariance matrix needs to be approximated numerically (in which case KL expansion is in fact the principle component analysis) and numerical calculation of eigen-quantities is often unstable. Second, a separate set of eigen-quantities needs to be numerically calculated for each bit configuration. (For a good discussion, see [13]; further details can be found in [7], [11], and [3]).

In this letter, we use a method of optimal fluctuations, or *instantons*, to model far tails, and numerical simulations to refine the middle part of pdfs. We combine these two approaches by using Edgeworth expansion with orthogonal polynomials specially constructed for this problem.

Edgeworth expansion is a statistical method for approximating unknown pdfs [16]. An unknown distribution $w(x)$ is approximated by successive improvements of a known starting approximation $u(x)$ by numerically (or experimentally) obtained moments of $w(x)$. In the existing literature, this method is almost always used with a Gaussian distribution as $u(x)$ [16], probably because in this case it involves the widely known Hermite polynomials (this special case of Edgeworth expansion is referred to as Gram–Charlier expansion). Recently, use of Gram–Charlier expansion was suggested in the context of optical communication systems [17]; this letter can be seen as a further elaboration of that work.

In this work, we use pdfs derived by the instanton method as the starting approximation $u(x)$. By using these pdfs, we obtain better asymptotic properties of the approximate distributions, which leads to faster and more accurate approximation of unknown pdfs. This requires derivation of a special family of orthogonal polynomials, which is given in Section III.

In Section IV, we apply the proposed method to find pdfs for a single-channel system with parameters corresponding to submarine systems.

II. PDFs Obtained By Instanton Method

The derivation of the instanton approximation for the pdfs follows the method in [5]. It is similar to the numerical saddle point approach used in [12] (an instanton is a saddle point in a functional space, see [5]), but in contrast to [12], we calculate

Manuscript received October 10, 2006; revised June 6, 2007. This work was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy (DOE) at Los Alamos National Laboratory under Contract DE-AC52-06NA25396. It was also supported in part by the National Science Foundation (NSF) under Grant ITR 0325979.

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Digital Object Identifier 10.1109/LPT.2007.904909

the saddle point, i.e, "the most damaging" noise configuration analytically.

The derivation is done under the assumption that noise from the amplifiers is Gaussian with zero mean and is uncorrelated (time span in which noise from erbium-doped fiber amplifiers (EDFAs) is correlated is short even compared with the duration of the bit slot [8]). The nonlinear interaction is also neglected, so this method (without a higher order correction, such as one presented in Section III) gives a pdf approximation that is applicable only for propagation distances below the nonlinear distance of a considered system.

Under the assumptions above, the pdf for energy at the center slot of bit pattern s is

$$\begin{aligned} u(E|s) &= \left\langle \delta\left(E - \int_{-T/2}^{T/2} |A_s(t,z)+\vartheta(t,z)|^2\,dt\right)\right\rangle_\vartheta \\ &= \int_{-\infty}^{\infty} d\lambda \int D\vartheta \exp\left(-\frac{1}{2Nz}\int_{-T/2}^{T/2}|\vartheta|^2 dt\right) \\ &\quad \times \exp\left(-i\lambda\left(E - \int_{-T/2}^{T/2}|A_s(t,z)+\vartheta(t,z)|^2\,dt\right)\right) \end{aligned} \tag{1}$$

where T is the bit duration and s is a bit configuration surrounding the center bit, which gives $A_s(t,z)$ via nonlinear interaction of the pulses in the absence of noise, and δ stands for the delta function, which, after the second equals sign, is realized via the integral with respect to λ. The averaging $\langle\rangle_\vartheta$ is done over all possible pulses coming from noise ϑ by the Feynman path integral $D\vartheta$.

This integral can be estimated by evaluating it around its saddle point(s) (instantons). Denoting $\Phi = (-(1/2Nz)\int_{-T/2}^{T/2}|\vartheta|^2 dt - i\lambda(E - \int_{-T/2}^{T/2}|A_s+\vartheta|^2 dt))$, we have: $\delta\Phi/\delta\vartheta = 0$, $\delta\Phi/\delta\vartheta^* = 0$, $\partial\Phi/\partial\lambda = 0$, where in this case, δ is the functional derivative. This system has only one solution for ϑ—the instanton. Using this solution in (1), and some algebra, we get the final expression for the pdf of energy to be

$$u(E|s) \approx C\exp\left[-\frac{1}{2Nz}\left(\sqrt{E} - \sqrt{\int_{-T/2}^{T/2}|A_s(t,z)|^2}\right)^2\right] \tag{2}$$

where C is the normalization constant.

We note that this expression is similar to the first-order pdf approximation obtained in [14]. Considering that the time-bandwidth product in modern systems is between 3 and 5 [3], the instanton approximation is consistent with the approximation in [14].

III. Edgeworth Expansion

In this section, we use Edgeworth expansion [16] to refine pdfs from Section II.

An unknown distribution $w(x)$ can be represented as

$$w(x) = u(x)\left[\sum_{i=1}^{\infty} C_i R_i(x)\right] \tag{3}$$

where $u(x)$ is a starting approximate distribution, and $R_i(x)$, $n \in N$ is a family of polynomials orthogonal with respect to the weight $u(x)$. Let $R_i(x) = \sum_{k=1}^{i} a_k x^k$. By multiplying (3) by $R_i(x)$ and integrating over the domain of orthogonality (in our case $x \in (0,+\infty)$, since x represents energy), we get $\sum_{k=1}^{i} a_k\eta_k = C_i$, where η_k and $k \in N$ are moments of the distribution $w(x)$. We can obtain a finite number j of these moments numerically or experimentally, therefore, deriving an approximate pdf $\tilde{w}_j(x)$ by truncating the infinite sum (3) to j terms. These approximate pdfs $\tilde{w}_j(x)$, $j \in N$, are guaranteed to converge to $w(x)$ uniformly (the interested reader is referred to [16] for a detailed exposition of the method).

In order to use the pdfs given in (2) as starting distributions, we need polynomials $R_n(x;m;p)$, $n \in N$, orthogonal with respect to the weight

$$u(x;m;p) = e^{-m(\sqrt{x}-p)^2}, \quad x \in (0,+\infty) \quad (m,p>0).$$

These polynomials can be seen as a generalization of Laguerre polynomials and to the best of our knowledge have not been studied before. We shall briefly explain how to construct them; their properties will be studied in a separate publication. The moments of the distribution $u(x;m;p)$ can be written as

$$\mu_n(m;p) = \frac{\sigma_{2n+1}(p)}{m^{2n+2}}$$

where

$$\sigma_n(p) = 2\int_{-p}^{+\infty}(t+p)^n e^{-t^2}dt.$$

Knowing all the moments $\mu_n(m;p)$, we can calculate the polynomials by

$$R(x;m;p) = \frac{\det(E_n)}{m^{(n+1)(3n-2)}\sigma_1^n(p)} \tag{4}$$

where the matrix $E_n = [e_{i,j}]_{(n+1)\times(n+1)}$ has elements $e_{i,j} = m^{2(n-j)}\mu_{i+j-1}$, $(j = 1,2,\ldots,n)$, $e_{i,n+1} = m^{2(n+i)-4}x^{i-1}$, $(i = 1,2,\ldots,n+1)$.

IV. Numerical Results

In this section, we shall illustrate the method. We considered the system that consists of periodically distributed sections of fiber with positive and negative dispersion separated by amplifiers (EDFA). One *span* consist of a section of fiber with positive dispersion, a section of fiber with negative dispersion, and corresponding amplifiers. The length of a span is set to 50 km.

The transmission of a signal through the fiber is modeled by the nonlinear Schrödinger equation (NLSE) [2]. In the system simulator, propagation of pulses through the system, i.e., solving NLSE, was done numerically by the split-step Fourier method.

The parameters of fibers are given in [6]. A precompensation of -330 ps/nm and a corresponding postcompensation were applied. The return-to-zero modulation format has a duty cycle of 33%, and the launched power was set to 0 dBm. EDFAs with noise figure of 8 dB were deployed after every span.

The nonlinear distance of this system is roughly 6000 km. For distances below this number, the instanton approximation itself approximates the true pdf well. To illustrate this, in Fig. 1, we plot both pdfs obtained by the instanton method and numerically obtained data for two bit configurations: 1) with zero in the center slot $s = 0110110$, and 2) with "1" at the center slot $s = 0001000$. Both bit configurations were propagated through 100 spans, that is 5000 km. However, when the propagation distance is longer, the instanton approximation is not sufficient.

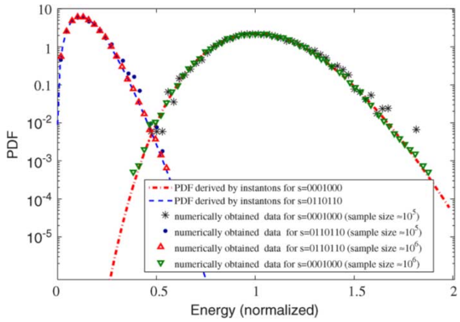

Fig. 1. PDFs after 100 spans. Normalization is chosen so that the peak of the pdf for the bit pattern "0001000" corresponds to 1.

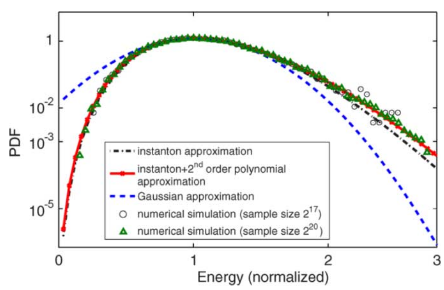

Fig. 2. Comparison of various pdf approximations for center bit energy within bit pattern "0001000" after 300 spans. Normalization is chosen so that the peak of the pdf corresponds to 1.

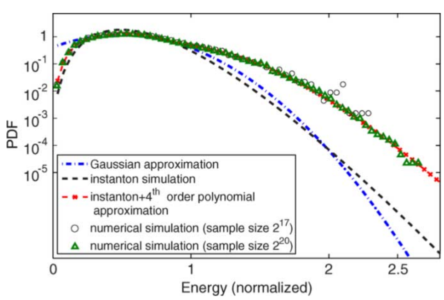

Fig. 3. Comparison of various pdf approximations for center bit energy within bit pattern "0110110" after 300 spans. Normalization is chosen so that 1 on the x-axis corresponds to 1 as in Fig. 2.

As Figs. 3 and 2 show, after 300 spans (approximately two and a half nonlinear distances) neglecting the nonlinear interaction between noise and pulses in the instanton approximation makes this approximation overly "optimistic," i.e., too narrow. After a polynomial correction is added, the refined pdfs differ very little from the numerically obtained data. We used 2^{17} numerically obtained samples to calculate the distribution moments needed for the polynomial correction. Further numerical simulations of length 2^{20}, shown with green triangles in Figs. 2 and 3, were not used to determine the curves, but show further agreement. Note that only polynomials up to fourth order are needed to refine the pdf for "0110110," and only the first two polynomials are needed for "0001000." This is in sharp difference from the result reported in [17], where, for a similar system, more than a dozen Hermite polynomials are needed to sufficiently improve the starting Gaussian distribution (also presented in Figs. 2 and 3).

A question regarding the number of polynomials needed to approximate an unknown distribution sufficiently well arises. From an engineering point of view, a satisfactory answer can be to add higher order polynomials until the moment difference between two consecutive refinements falls under a certain threshold.

V. Final Remarks

We developed an approach for approximating pdfs that is both practical and accurate. The instanton method gives suitable asymptotic behavior for the tails of distributions. Use of the parameterized family of orthogonal polynomials reduces computer processing cost, making this approach applicable for high-speed applications.

The method is also very general (it is not restricted to specific pulse shaping, bit rate, propagation distance) and, therefore, applicable to a wide range of systems.

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