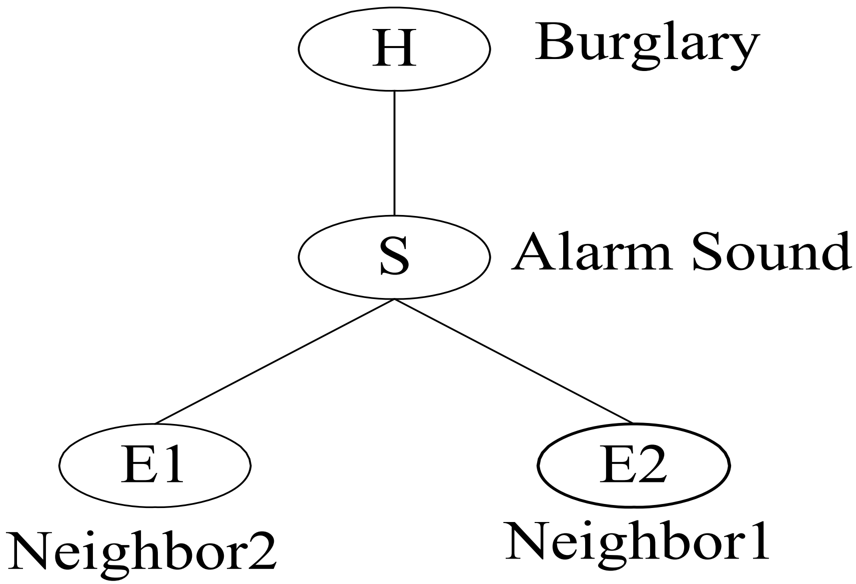
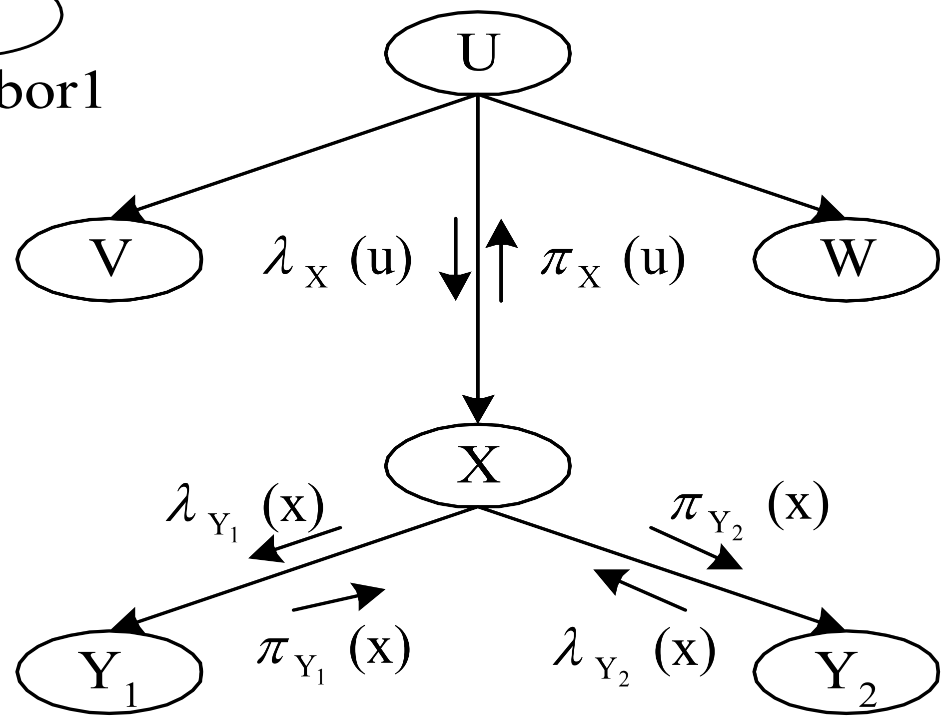
ECE 566

Bayesian Inference III


H
Burglary
S
Alarm Sound
E1
E2
Neighbor2
Neighbor1


U
V
W
$\lambda_X(u)$
$\pi_X(u)$
X
$\lambda_{Y_1}(x)$
$\pi_{Y_2}(x)$
$\pi_{Y_1}(x)$
$\lambda_{Y_2}(x)$
Y_1
Y_2

Initialization

1. Root Node: We set the $\pi(X)$ for a root node X to the prior probability for it : $\vec{P}(X)$

2. Anticipatory Nodes: If a node X is a childless node, we set $\lambda(X) = [1\ 1\ \ldots\ 1]^{T}$.

3. Evidence Nodes: They don't receive π messages. They provide a λ vector. The λ node value is set to the likelihoods the evidence provides for each possible value of their parent node.

Belief Propagation and Belief Accumulation

Step 1: Belief Update

When node X is activated, it updates its own parameters, λ, π from λ supplied by its children and π supplied by its parent:

$$\lambda(\mathrm{X}) = \prod_{\mathrm{j}} \lambda_{\mathrm{Y_i}}(\mathrm{X}) \quad \text{term-wise product}$$

$$\pi(\mathrm{X}) = \mathrm{M}_{\mathrm{X|U}}\,\pi_{\mathrm{X}}(\mathrm{U})$$

$$\mathrm{Bel}(\mathrm{X}) = \alpha\lambda(\mathrm{X})\pi(\mathrm{X})$$

α: normalizing parameter to make sure $\sum_{\mathrm{X_i}} \mathrm{Bel}(\mathrm{X} = \mathrm{X_i}) = 1$

Step 2: Bottom-up Propagation

Using the λ messages received, node X computes a new λ message, $\lambda_X(U)$ which it sends to its parent U.

$$\lambda_X(U) = \lambda(X)\, M_{X|U}$$

$$= \sum_{X=X_i} \lambda(X)\, P(X = X_i \mid U)$$

Step 3: Top down Propagation

Node X after updating itself, sends new π messages to each of its children:

$$\pi_{Y_j}(X) = \alpha \pi(X) \prod_{k \neq j} \lambda_{Y_k}(X)$$

$$= \frac{Bel(X)}{\lambda_{Y_j}(X)}$$

Example

A murder has occurred and there are three suspects. The fingerprint lab provides information about fingerprints on the murder weapon.

Let X, and Y represent the following hypotheses:

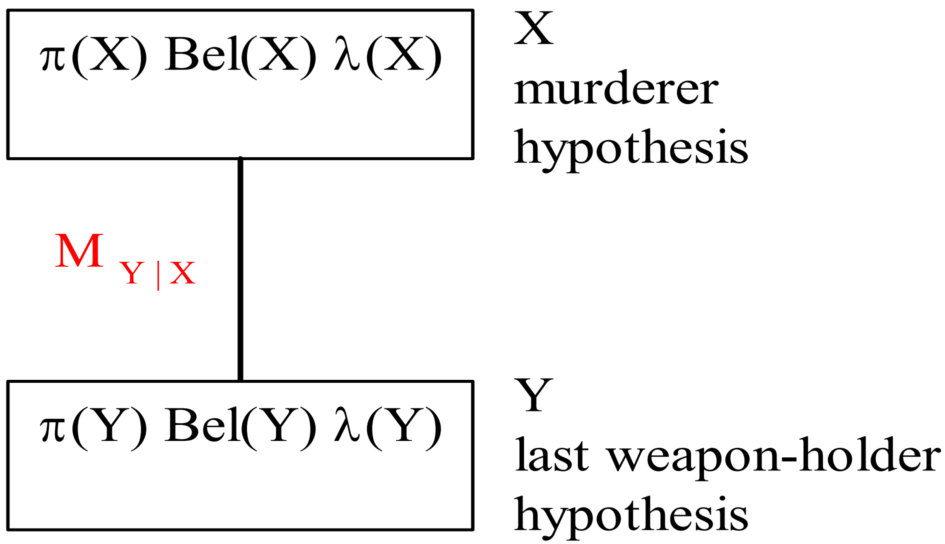
X: identity of the killer

Y: identity of the person who last held the weapon

Assuming that the conditional probability matrix $M_{Y|X}$ is as shown and the prior probability for each suspect is $\vec{P}(X) = [0.8\ 0.1\ 0.8]^T$

$$M_{Y|X} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

a) Show the belief network depicting this system and use the given information to initialize its parameters.



Initializing:

$$\pi(X) = P(X) = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix}$$

$$\lambda(Y) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{anticipatory node}$$

$$\lambda(\mathrm{X}) = \frac{\mathrm{Bel(X)}}{\pi(\mathrm{X})} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\pi(\mathrm{Y}) = \pi_{\mathrm{X}}(\mathrm{Y})\ \mathrm{M}_{\mathrm{Y|X}}$$

$$= \begin{bmatrix} 0.8 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix}$$

$$\mathrm{Bel(Y)} = \alpha\pi(\mathrm{Y})\ \lambda(\mathrm{Y}) = \alpha \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix}$$

$$(\alpha = 1)$$

X

$$\pi(X) = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} \quad Bel(X) = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} \quad \lambda(X) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$M_{Y|X} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Y

$$\pi(Y) = \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix} \quad Bel(Y) = \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix} \quad \lambda(Y) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

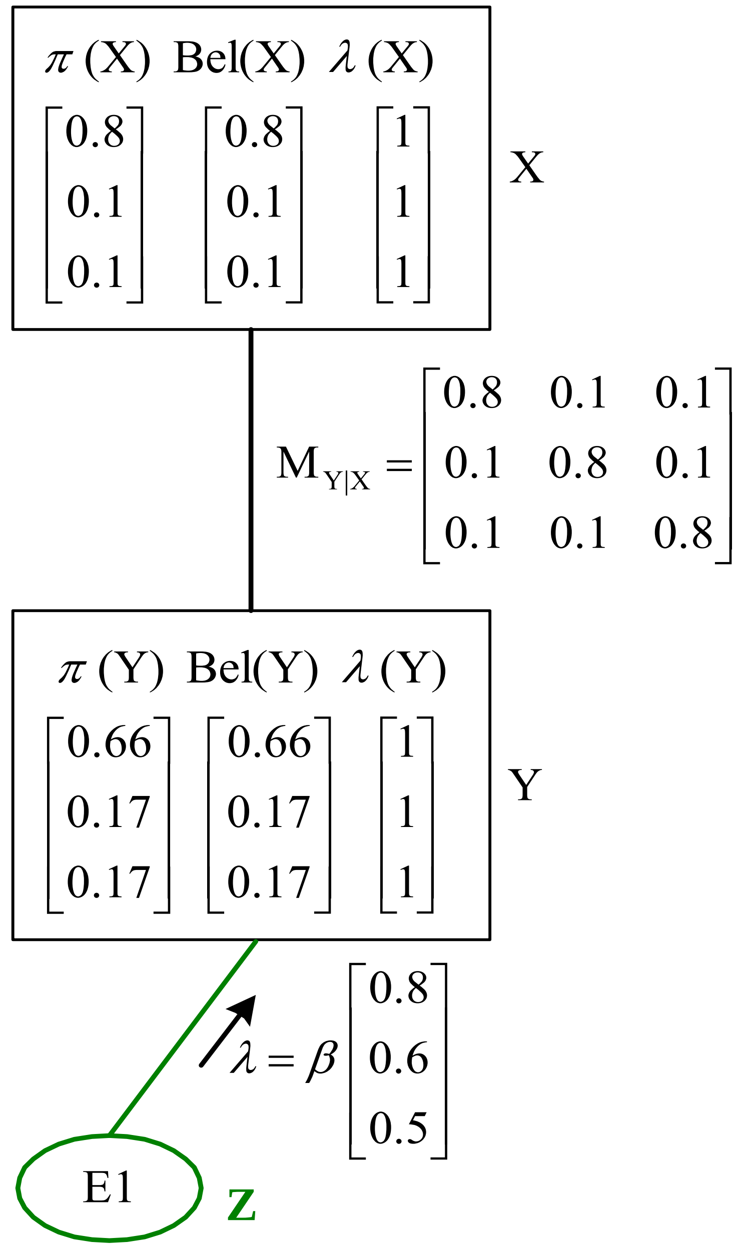
b) Assume the information (evidence) becomes available about fingerprints on the weapon. The evidence states the relative likelihood as $[0.8\ 0.6\ 0.5]^T$.

Propagate the evidence through the network and calculate the updated beliefs.

Let Z be the node representing the result from the fingerprint lab.

Z: an evidence node

Y receives a message from
Z (a lab report) and updates itself.



$$\lambda_Z(Y) = \beta \begin{bmatrix} 0.8 \\ 0.6 \\ 0.5 \end{bmatrix}$$

Updating node Y :

$$\lambda(Y) = \prod_i \lambda\ Y_i \qquad \text{where } Y_i \text{ are children}$$

$$\lambda(Y) = \beta \begin{bmatrix} 0.8 \\ 0.6 \\ 0.5 \end{bmatrix}$$

$$\pi(Y) = \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix}$$

$$\Rightarrow \mathrm{Bel}(Y) = \alpha\, \pi(Y)\, \lambda(Y)$$

$$\mathrm{Bel}(Y) = \alpha \underbrace{\begin{bmatrix} 0.8 \\ 0.6 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.66 \\ 0.17 \\ 0.17 \end{bmatrix}} = \begin{bmatrix} 0.738 \\ 0.143 \\ 0.119 \end{bmatrix}$$

$$\begin{bmatrix} 0.528 \\ 0.102 \\ 0.085 \end{bmatrix} \Rightarrow \alpha = \frac{1}{0.528 + 0.102 + 0.085} = \frac{1}{0.715}$$

Bottom-up propagation to X:

$$\lambda_Y(X) = M_{Y|X}\lambda(Y) = \beta \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.6 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \\ 0.61 \\ 0.54 \end{bmatrix}$$

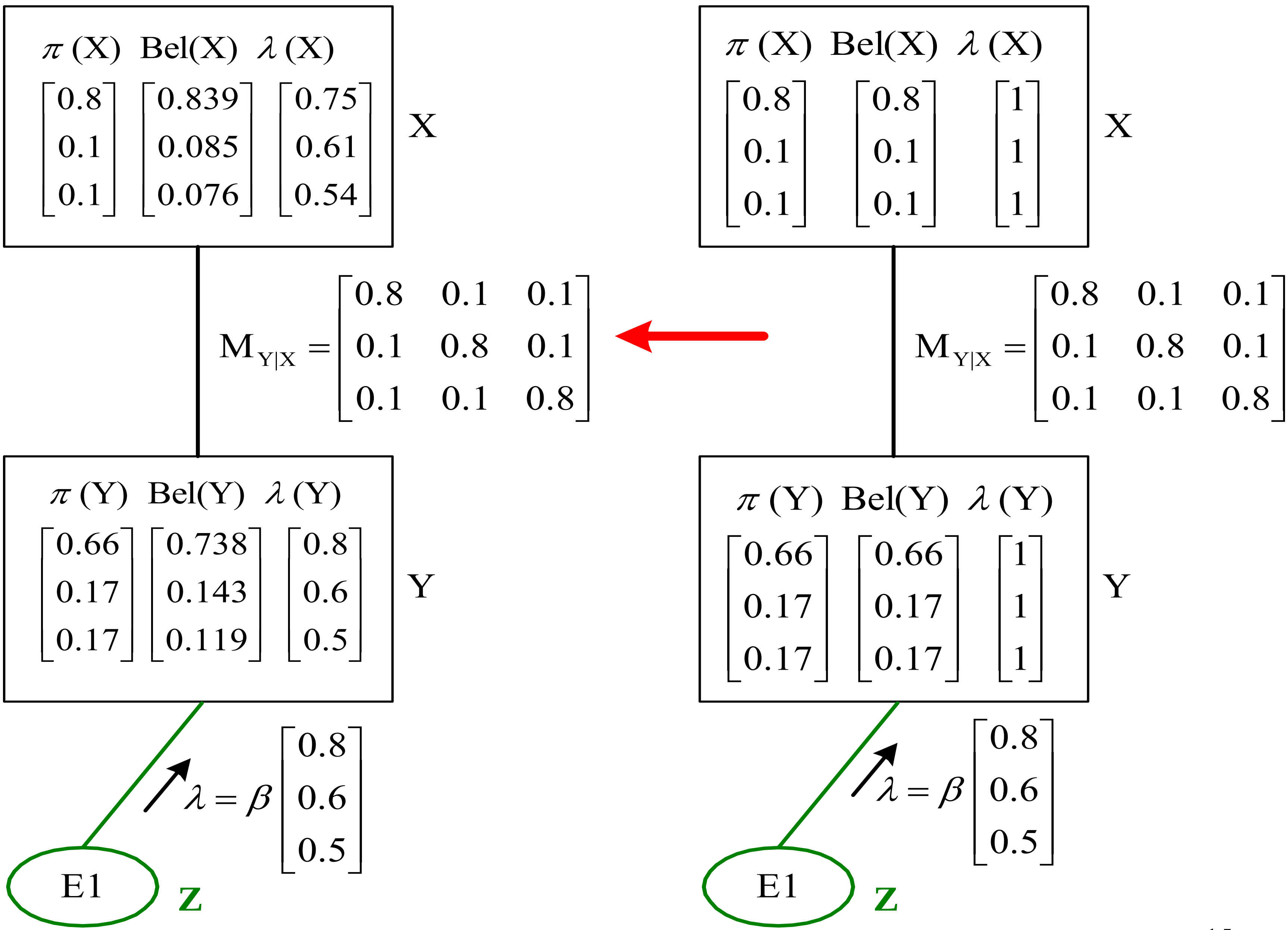
$$\lambda(X) = \prod_i \lambda_{Y_i}(X)$$

$$= \begin{bmatrix} 0.75 \\ 0.61 \\ 0.54 \end{bmatrix}$$

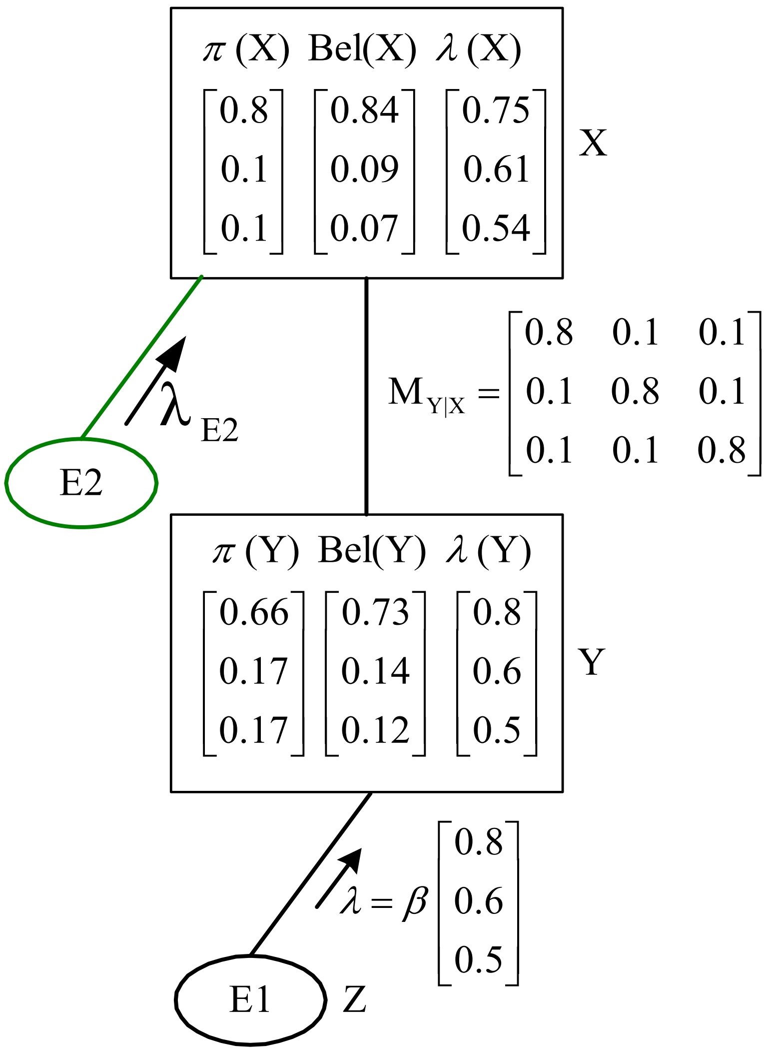
$$\pi(X) = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix}$$

$$\mathrm{Bel}(X) = \alpha\pi(X)\lambda(X)$$

$$= \alpha \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.61 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 0.839 \\ 0.085 \\ 0.076 \end{bmatrix}$$


π (X) Bel(X) λ (X)
0.8 0.1 0.1
0.839 0.085 0.076
0.75 0.61 0.54
X
M Y|X =
0.8 0.1 0.1
0.1 0.8 0.1
0.1 0.1 0.8
π (Y) Bel(Y) λ (Y)
0.66 0.17 0.17
0.738 0.143 0.119
0.8 0.6 0.5
Y
λ = β
0.8 0.6 0.5
E1
Z
π (X) Bel(X) λ (X)
0.8 0.1 0.1
0.8 0.1 0.1
1 1 1
X
M Y|X =
0.8 0.1 0.1
0.1 0.8 0.1
0.1 0.1 0.8
π (Y) Bel(Y) λ (Y)
0.66 0.17 0.17
0.66 0.17 0.17
1 1 1
Y
λ = β
0.8 0.6 0.5
E1
Z

c) New information becomes available. Suspect 1 produces a strong alibi. Assume that this information provides the likelihoods β_2[1 10 10] for identity of the killer. Integrate this information into the belief network and update the beliefs.



$$\lambda_{E2}(X) = \beta_2 \begin{bmatrix} 1 \\ 10 \\ 10 \end{bmatrix}$$

$$\lambda(X) = \lambda_{E2}(X)\lambda_Y(X) = \beta_2 \begin{bmatrix} 1 \\ 10 \\ 10 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.61 \\ 0.54 \end{bmatrix} = \beta_2 \begin{bmatrix} 0.75 \\ 6.1 \\ 5.4 \end{bmatrix}$$

$$\begin{aligned} \text{Bel}(X) &= \alpha\, \lambda(X)\pi(X) \\ &= \alpha \begin{bmatrix} 0.75 \\ 6.1 \\ 5.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} \\ &= \begin{bmatrix} 0.343 \\ 0.349 \\ 0.308 \end{bmatrix} \end{aligned}$$

Top-down propagation to Y:

$$\pi_{Y}(X) = \alpha\pi(X)\prod_{k\neq y}\lambda_k$$

$$\pi_{Y}(X) = \alpha\begin{bmatrix}0.8\\0.1\\0.1\end{bmatrix}\begin{bmatrix}1\\10\\10\end{bmatrix} = \alpha\begin{bmatrix}0.8\\1\\1\end{bmatrix}$$

$$\pi(Y) = \pi_{Y(X)} M_{Y|X}$$

$$= \alpha\begin{bmatrix}0.8 & 1 & 1\end{bmatrix}\begin{bmatrix}0.8 & 0.1 & 0.1\\0.1 & 0.8 & 0.1\\0.1 & 0.1 & 0.8\end{bmatrix}$$

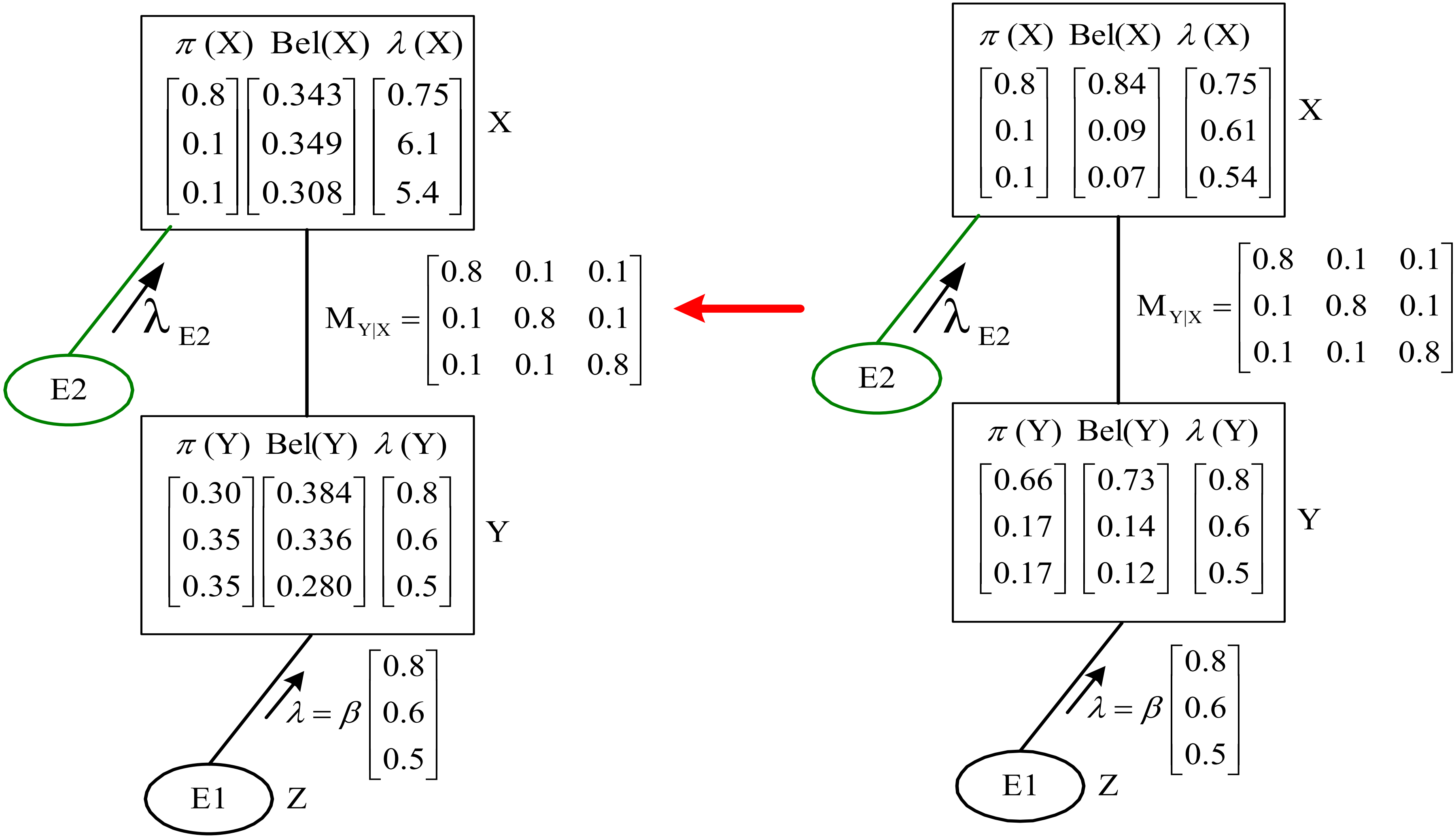
$$= \begin{bmatrix}0.30\\0.35\\0.35\end{bmatrix}$$

$$\lambda(Y) = \begin{bmatrix}0.8\\0.6\\0.5\end{bmatrix}$$

$$\text{Bel}(Y) = \alpha\pi(Y)\lambda(Y)$$

$$= \alpha\begin{bmatrix}0.30\\0.35\\0.35\end{bmatrix}\begin{bmatrix}0.8\\0.6\\0.5\end{bmatrix}$$

$$= \begin{bmatrix}0.384\\0.336\\0.280\end{bmatrix}$$



The new information reduces the likelihood that the murderer is suspect 1 from 0.84 to 0.343.