2-D FILTER DESIGN

TYPES

- *Finite Impulse Response (FIR) – described here*
- *Infinite Impulse Response (IIR) – not discussed*

ZERO-PHASE FILTERS

- ***h(m,n) is zero-phase if*** $H(k, l) = H^*(k, l)$***, i.e. H is real***
- ***Zero-phase desired in digital filters to avoid signal distortion***

  Zero-phase not guaranteed in physical filters, e.g. optical defocus and square scan spots (see Notes4 and 5)
- ***H real implies***

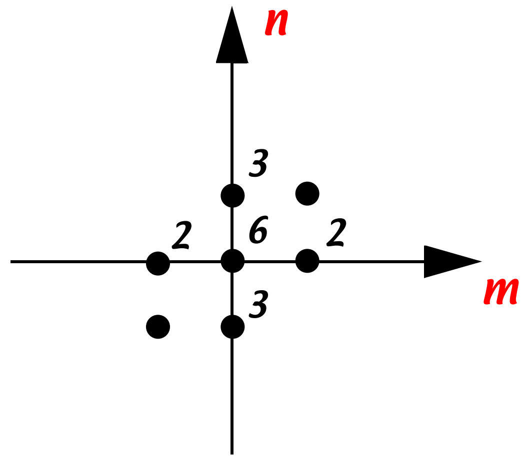
$$h(m, n) = h^*(-m, -n)$$

Prove the above equation

2-D FILTER DESIGN

- *Furthermore, since* ***h is assumed real*** *in our case,*

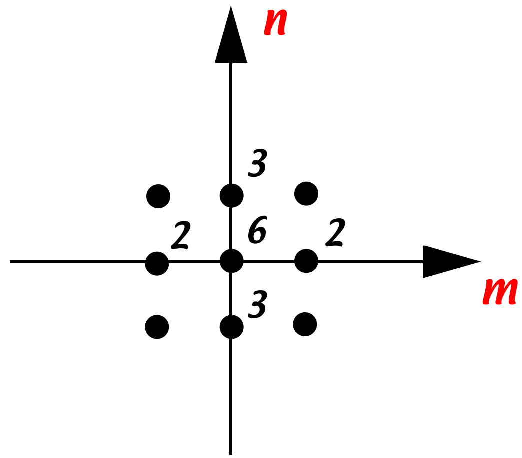
$h(m, n) = h(-m, -n)$, *i.e.* ***h*** *is* ***two-fold*** *symmetric*



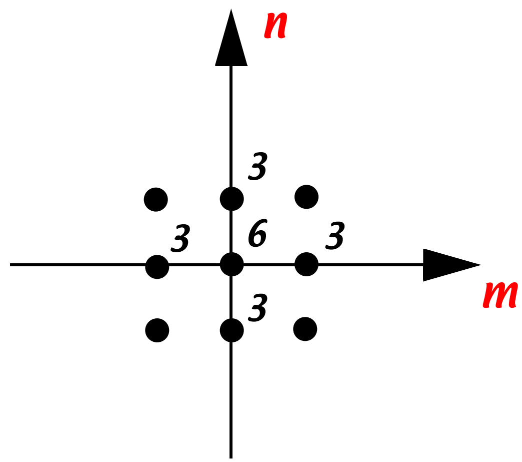
- *Greater symmetry reduces number of independent points (Degrees of Freedom (DOF)) required in filter design*

- ***Additional symmetries***

***four-fold* (includes two-fold):** $h(m, n) = h(-m, n) = h(m, -n)$



***eight-fold* (includes four-fold):** $h(m, n) = h(-m, n) = h(m, -n) = h(n, m)$



FILTER SPECIFICATION

- *Frequency domain*
- *Ideal, continuous, radial filters (note, none are FIR)*

type	$F(\rho)$	$f(r)$
Low-Pass FIlter (LPF)	$cyl\left(\frac{\rho}{2\rho_c}\right)$	$\frac{\pi}{4} somb(2\rho_c r)$
High-Pass Filter (HPF)	$1 - cyl\left(\frac{\rho}{2\rho_c}\right)$	$\frac{\delta(r)}{\pi r} - \frac{\pi}{4} somb(2\rho_c r)$
Band-Pass Filter (BPF)	$cyl\left(\frac{\rho}{2\rho_2}\right) - cyl\left(\frac{\rho}{2\rho_1}\right),\ \rho_2 > \rho_1$	$\frac{\pi}{4}[somb(2\rho_2 r_2) - somb(2\rho_1 r_1)]$
Band-Stop Filter (BSF)	$1 - cyl\left(\frac{\rho}{2\rho_2}\right) + cyl\left(\frac{\rho}{2\rho_1}\right),\ \rho_2 > \rho_1$	$\frac{\delta(r)}{\pi r} - \frac{\pi}{4}[somb(2\rho_2 r_2) - somb(2\rho_2 r_2)]$

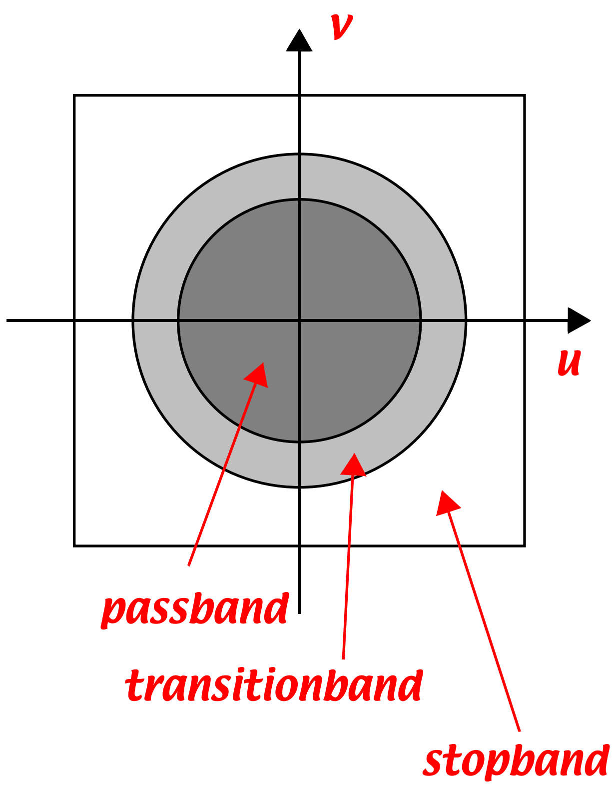
2-D FILTER DESIGN

- *Amplitude transition cannot be sharp with real filters*
- *Therefore, specify transition region and tolerances in designed filters*

in passband region: $1 - \varepsilon_p < |H(u, v)| < 1 + \varepsilon_p$

in stopband region: $|H(u, v)| < \varepsilon_s$

which measure quality of filter relative to ideal case



FILTER DESIGN

- ***Assume a continuous, periodic frequency domain filter and a discrete spatial domain filter, with the Discrete-Space Fourier Transform relation,***

$$H(u,v) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h(m,n)e^{-j2\pi um}e^{-j2\pi vn}$$

$$h(m,n) = \int_{u=-1/2}^{1/2}\int_{v=-1/2}^{1/2} H(u,v)e^{j2\pi um}e^{j2\pi vn}\,du\,dv$$

Rewrite in terms of angular frequencies $\omega_1 = 2\pi u$, $\omega_2 = 2\pi v$

Examples

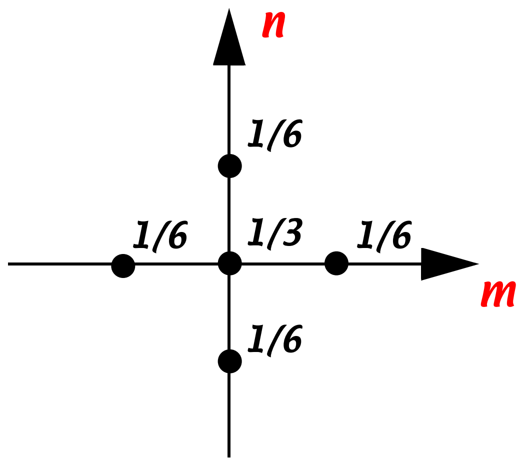
- ***Specify H and calculate h***

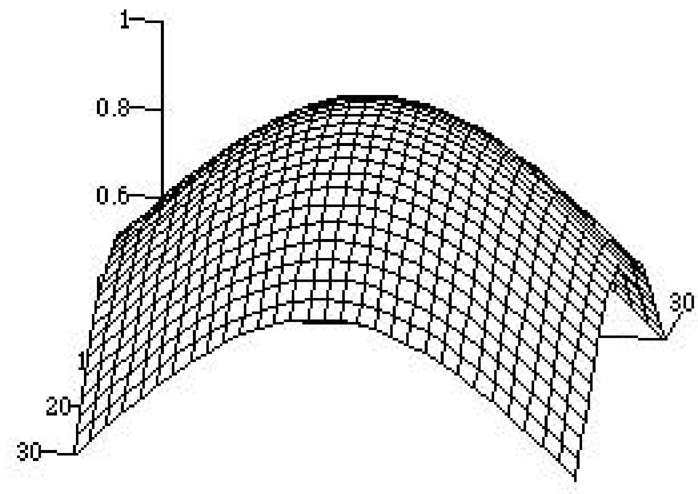
$$H(u,v) = rect\left(\frac{u}{a}, \frac{v}{b}\right), |u| \le 1/2, |v| < 1/2$$

$$h(m,n) = |a||b|\,\mathrm{sinc}(am, bn)$$

2-D FILTER DESIGN

- ***Specify h and calculate H***





$$
\begin{aligned}
H(u, v) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n) e^{-j2\pi um} e^{-j2\pi vn} \\
&= \frac{1}{3} + \frac{1}{6}(e^{-ju} + e^{ju} + e^{-jv} + e^{jv}) \\
&= \frac{1}{3} + \frac{1}{6}(2\cos u + 2\cos v) \\
&= \frac{1}{3}(1 + \cos u + \cos v)
\end{aligned}
$$

2-D FILTER DESIGN

Methods for Filter Design

- window (1-D extension)
- frequency sampling (1-D extension)
- transformation (unique to 2-D)

2-D FILTER DESIGN

Window Method

- **Assume a desired frequency response** $H_d(u, v)$
- **Ideal frequency domain filters result in Infinite Impulse Responses (IIR)**
- **Limit the extent of the spatial response with a window function w**

$$h(m, n) = h_d(m, n)w(m, n)$$

- **Corresponds to a convolution in frequency domain**

$$H(u, v) = H_d(u, v) \ast\ast W(u, v)$$

- **Desirable properties of W(u,v):**
  - **narrow main-lobe**
  - **low side-lobes**
  - **finite extent in spatial domain**

2-D FILTER DESIGN

Generation of 2-D windows from 1-D windows

- **Two ways:**

Implement 1-D window functions in separable 2-D window

$$w(m,n) = w_1(m)w_2(n)$$

$$W(u,v) = W_1(u)W_2(v)$$

Rotate 1-D window about the origin as "generating function" to create 2-D radial (circularly symmetric) function (see Notes2)

$$w(m,n) = w(\sqrt{m^2+n^2})$$

$$W(u,v) = W(\sqrt{u^2+v^2})$$

2-D FILTER DESIGN

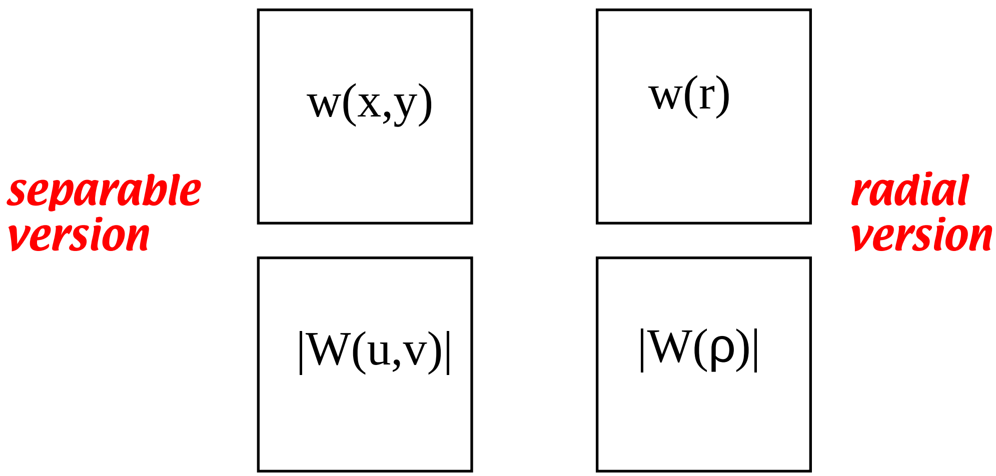
- *Examples*
- *All defined with:*

width b=8

plot array M=N=32

spatial region of support = 8 x 8 (separable) or radius = 4 (radial)

plot layout:

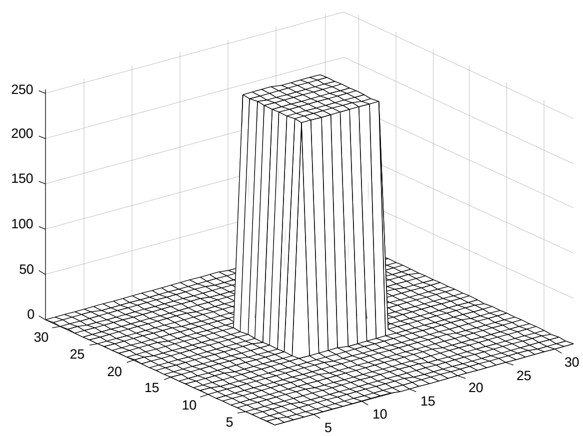


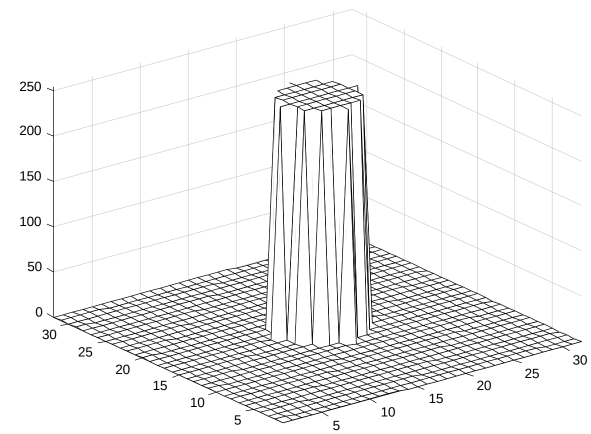
2-D FILTER DESIGN

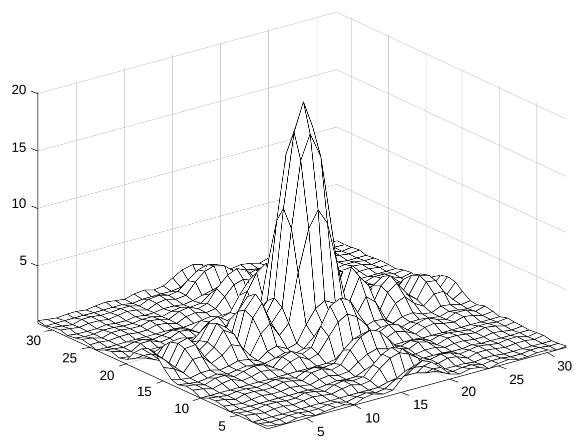
- ***rectangle*** $w(x) = 1$ ***(see Notes1)***

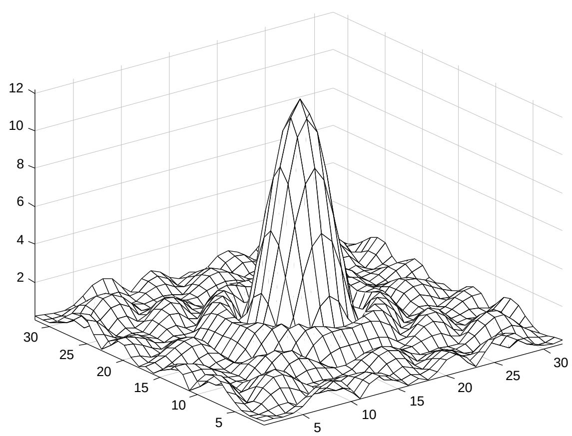
- ***cylinder*** $w(r) = cyl\left(\frac{r}{b}\right)$

rectangle window - separable version ***radial version***



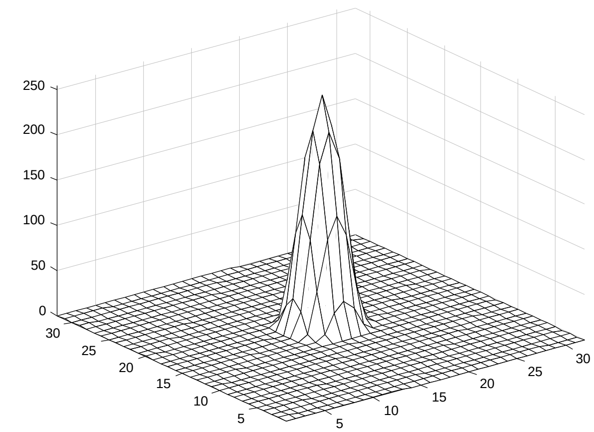


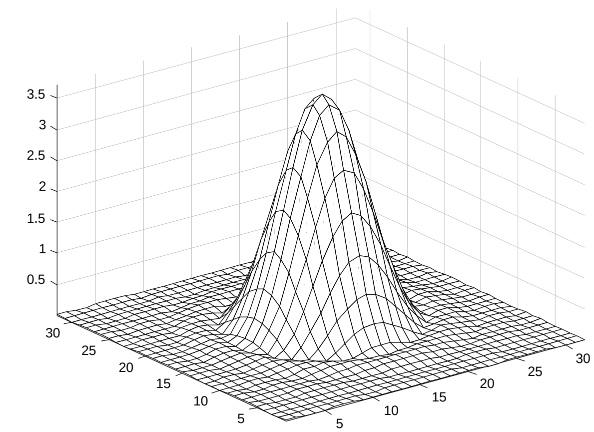




2-D FILTER DESIGN

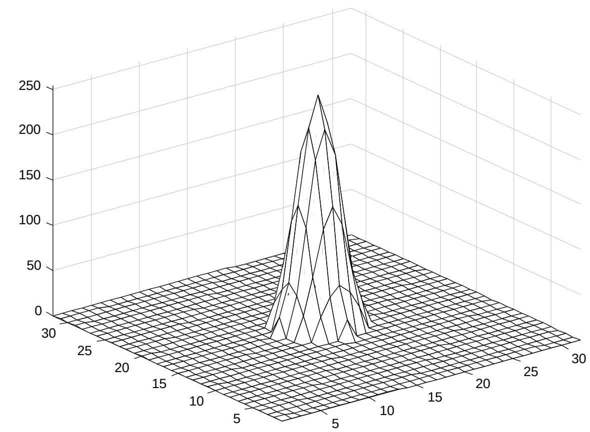
- ***Hanning (raised cosine)*** $w(r) = 0.5 + 0.5\cos\left(2\pi \frac{r}{b}\right)$

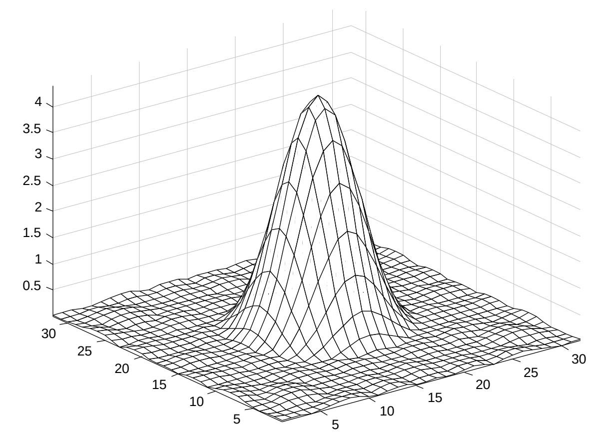




2-D FILTER DESIGN

- **Hamming** $w(r) = 0.54 + 0.46\cos\left(2\pi\frac{r}{b}\right)$



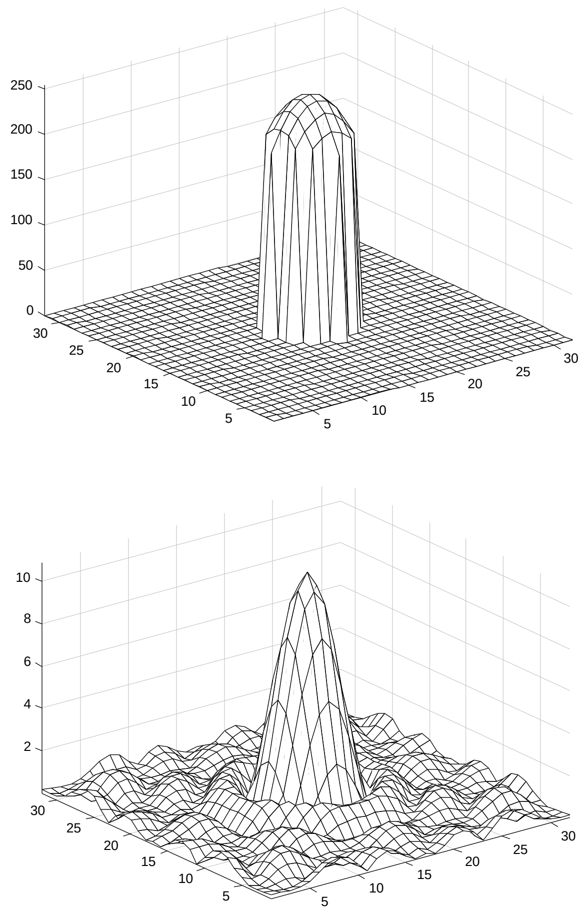


2-D FILTER DESIGN

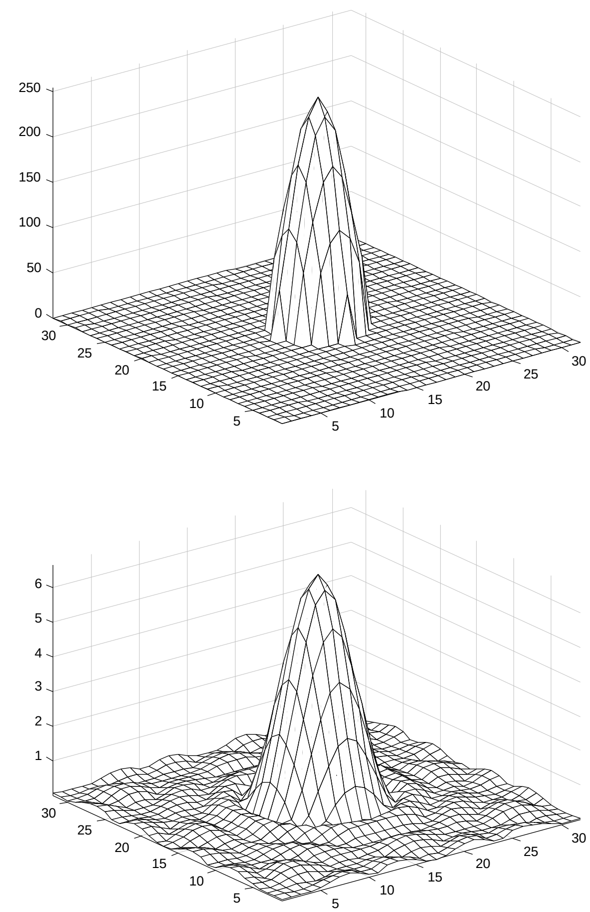
- **Kaiser** $w(r) = \frac{1}{I_0(\alpha)} I_0\left(\alpha\sqrt{1-(2r/b)^2}\right)$

where I_0 is a modified Bessel function of the first kind, order zero, and α controls the window shape: α smaller –> mainlobe narrower, α larger –> sidelobes lower

α = 1 radial version



α = 3



2-D FILTER DESIGN

Frequency Sampling Method

- Sample 2-D continuous frequency domain filter H(u,v) to obtain H(k,l) (origin-centered)
- Calculate inverse DFT to obtain h(m,n) (origin-centered)

Frequency Sampling Method

- add linear phase (M and N odd)

$$H_l(u, v) = H_z(u, v)e^{-j2\pi u((M-1)/2)}e^{-j2\pi v((N-1)/2)}$$

- sample

$$H_l(k, l) = H_l(k/M, l/N)$$
$$k = 0, 1, \quad M-1$$
$$l = 0, 1, \quad N-1$$

- inverse DFT $h_l(m, n)$

- shift $h_z(m, n) = h_l(m + (M-1)/2, n + (N-1)/2)$

2-D FILTER DESIGN

- *Sampling of ideal frequency domain filters results in discontinuities at cut-off and cut-on frequencies*

- *Apply "smoothing" in transition region*

  Simple linear transition works fine

- *Origin-centering important to maintain zero phase*