ECE 304: Open- and Short-circuit Time Constants
See S&S, pp. 575-578, 497-503

Example Circuit

PARAMETERS:
\[ R_1 = \{R_L\} \quad C_1 = \{C_{LF}\} \quad C_{LF} = 1\mu F \quad \pi = 3.1415926 \]
\[ R_2 = \{R_H\} \quad C_2 = \{C_{LF}\} \quad C_{HF} = 1pF \]
\[ R_3 = \{R_L\} \quad C_3 = \{C_{HF}\} \quad R_H = \{(1E3/(2*\pi))\} \]
\[ R_4 = \{R_H\} \quad C_4 = \{C_{HF}\} \quad R_L = \{(10/(2*\pi))\} \]

**FIGURE 1**
An RC-circuit that is a midband filter.

The first two sections in Figure 1 are high-pass filters and the last two are low-pass filters. The factor \(2\pi\) is introduced to make the frequencies easier to estimate.

**FIGURE 2**
Bode magnitude plot for circuit of Figure 1. Midband region lies between the lower 3dB-corner frequency of 102 kHz and the upper 3dB-corner frequency of 980 MHz; midband gain is approximately zero dB

**FIGURE 3**
Bode phase plot with phases marked at the pole frequencies for each section. The phases are slightly affected by the nearness (proximity) of other poles. For isolated poles, the phases at the pole frequencies taking midband as 0° are 135°, 45°, −45°, and −135°
Midband Analysis
At midband, all low-frequency capacitors are short-circuits and all high-frequency capacitors are open circuits. That is, C₁ and C₂ are shorts, and C₃, C₄ are opens, making the gain \( \frac{V_{OUT}}{V_{IN}} = 1 \) V/V or 0 dB.

High-frequency analysis
At high frequencies, near or above the upper corner frequency of 980 MHz, we can approximate the two low-frequency capacitors C₁ and C₂ by short circuits. The result is shown in Figure 4: the high-frequency gain of the two circuits is the same, but the low-frequency behavior is not, because C₁ and C₂ are not present in the approximate circuit.

![Figure 4]
Comparison of high-frequency approximation to circuit (capacitors C₁ and C₂ replaced by short circuits) and the original circuit of Figure 1

![Figure 5]
The approximate circuit valid at high-frequency

Circuit analysis of Figure 5 shows the gain to be

\[
\frac{V_{OUT}}{V_{S}} = \frac{1}{1 + j\omega\left[C_4 (R_3 + R_4) + C_3 R_3\right] + (j\omega)^2 C_4 R_3 R_4}.
\]

The dominant pole approximation
The denominator of EQ. 1 is quadratic in \( \omega \), which means there are two poles in the gain function and two high-frequency break points. In Figure 3, these occur at 980 MHz and 102 GHz, for this example. In the case that these two poles are well separated (the dominant pole case) the linear term in \( \omega \) determines the corner frequency that marks the end of the midband region.

If the term linear in \( \omega \) is examined more carefully, it is composed of two RC time constants, one related to each of the two high-frequency capacitors. For example, the term \( C_4 (R_3 + R_4) \) uses the resistance seen between the nodes where \( C_4 \) is attached if the applied signal source is shorted out. This resistance can be found by putting a test current or voltage in place of \( C_4 \), shorting \( V_{in} \) and open-circuiting \( C_3 \) (see Figure 6).
**FIGURE 6**
Circuit for finding the Thevenin resistance $(R_3 + R_4) = 160.7 \, \Omega$ seen by $C_4$

In Figure 6 a test current was used so we can read the resistance seen by $C_4$ simply as the voltage at the top of the test source. Likewise, the term $C_3R_3$ is found using Figure 7.

**FIGURE 7**
Circuit for finding the Thevenin resistance $R_3 = 1.59 \, \Omega$ seen by $C_3$

In Figure 7 a zero-current source is added at the end of $R_4$ because PSPICE does not allow dangling resistors. The zero-amp current source tells PSPICE that there is no current in $R_4$, as is appropriate when the circuit ends with an open circuit because $C_4$ has been replaced by an open circuit.

The general approach to finding the high-frequency corner is then as follows:

1. Draw the small-signal circuit
2. Short all the low-frequency capacitors
3. Short all independent AC voltage sources and open all independent AC current sources
4. Select a particular high-frequency capacitor. Replace all the others by open circuits.
5. Put a test voltage at the site of the selected capacitor and find the resistance seen at this site.
6. Multiply this resistance by the value of the selected capacitance $\rightarrow \tau_1$, say
7. Do the same for all the other high-frequency capacitances.
8. The high-frequency corner frequency is $f_C = 1/(2\pi) \times 1/(\tau_1 + \tau_2 + \tau_3 + \ldots)$

**Low-frequency analysis**

**FIGURE 8**
The approximate low-frequency circuit
Figure 9 shows the low-frequency circuit and the original circuit agree at low frequencies, but not at high frequencies, which is the reverse to Figure 4. Circuit analysis of Figure 8 shows the voltage gain is given by

\[
\frac{V_{\text{OUT}}}{V_s} = \frac{1}{1 + \frac{1}{j\omega} \left( \frac{1}{C_1(R_1/R_2)} + \frac{1}{C_2R_2} \right) + \frac{1}{(j\omega)^2} \left( \frac{1}{C_1C_2R_1R_2} \right)}
\]

**The dominant pole approximation**

In complete analogy with the high-frequency case, the denominator of EQ. 1 is quadratic in \(\omega\), which means there are two poles in the gain function and two low-frequency break points. In Figure 3, these occur at 980 Hz and 102 kHz, for this example. In the case that these two poles are well separated (the dominant pole case) the linear term in \(1/j\omega\) determines the corner frequency that marks the start of the midband region.

Looking at the linear term, it is composed of two RC time constants, one related to each of the two low-frequency capacitors. For example, the term \(C_1(R_1/R_2)\) uses the resistance seen between the nodes where \(C_1\) is attached if the applied signal source is shorted out. This resistance can be found by putting a test voltage or current in place of \(C_1\), shorting \(V_{sh}\) and short-circuiting \(C_2\) (see Figure 10).

![Circuit for finding the resistance \(R_1/R_2 = 1.58 \Omega\) seen from the site of \(C_1\)](image)

Likewise, the resistance seen by \(C_2\) is found using Figure 11.
The general approach to finding the low-frequency corner is then as follows:
1. Draw the small-signal circuit
2. Replace all the high-frequency capacitors with open circuits
3. Short all independent AC voltage sources and open all independent AC current sources
4. Select a particular low-frequency capacitor. Replace all the others by short circuits.
5. Put a test voltage at the site of the selected capacitor and find the resistance seen at this site.
6. Multiply this resistance by the value of the selected capacitance \( \tau_1 \), say
7. Do the same for all the other low-frequency capacitances.
8. The low-frequency corner frequency is \( f_C = \frac{1}{2\pi} \times \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + \ldots \right) \)

Finding the other pole

We've discussed finding the high- and low-frequency corner frequencies using the dominant pole approximation. How can the higher pole at 102 GHz (or the lower pole at 980 Hz) be found? A technique based on the dominant pole assumption is described next (see S&S, pp. 852-853).

The case of the high frequency pole will be discussed here. The low frequency case is similar.

We suppose that only two poles matter, so the denominator of the high-frequency gain expression is of the form

\[
D(j\omega) = (1 + j\omega\tau_1)(1 + j\omega\tau_2) = 1 + j\omega(\tau_1 + \tau_2) + (j\omega)^2\tau_1\tau_2
\]

In general, we won't have factored this expression so neatly; and we will have instead of EQ. 3

\[
D(j\omega) = 1 + j\omega a_1 + (j\omega)^2 a_2
\]

If one time constant, say \( \tau_1 \), is much bigger than the other, we can make the approximation

\[
\tau_1 \approx a_1,
\]

which means we can simply read \( \tau_1 \) as the coefficient of the linear term in \( j\omega \). Having made this approximation, the second time constant is simply

\[
\tau_2 = \frac{\tau_1\tau_2}{\tau_2} = a_2/\tau_2 \approx a_2/a_1,
\]

that is, the ratio of the coefficients. We can check after doing this to see if the assumption that \( \tau_1 \gg \tau_2 \) is accurate. How does this approach differ from a more exact approach? What is a more exact approach? Suppose we set \( s = j\omega \). Then, like EQ. 4,

\[
D(s) = 1 + s a_1 + s^2 a_2,
\]

which can be written like EQ. 3 in terms of its zeros (say, \( s = -1/\tau_1 \) or \( s = -1/\tau_2 \))
EQ. 8

\[ D(s) = (s+1/\tau_1)(s+1/\tau_2)\tau_1\tau_2 = 1 + s(\tau_1+\tau_2) + s^2(\tau_1\tau_2) \]

Using the quadratic formula, the zeros of \( D(s) \) (the poles of \( 1/D(s) \)) are given by

\[
s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2a_2} = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 + \tau_2)^2 - 4\tau_1\tau_2}}{2\tau_1\tau_2} = \frac{-(\tau_1 + \tau_2)}{2\tau_1\tau_2} \left( 1 \pm \frac{\sqrt{(\tau_1 - \tau_2)^2}}{(\tau_1 + \tau_2)^2} \right)
\]

If \( \tau_1 \gg \tau_2 \), the square root is nearly 1. For example, suppose it is 0.99. Then the bracketed term is either \( 1 + 0.99 \approx 2 \) or \( 1 - 0.99 = 0.01 \). In this last case, unless we have very good accuracy, the round-off error in this estimate for the shortest time constant will be large. Under these circumstances, EQ. 6 is more accurate than the quadratic formula. However, if \( \tau_1 \) and \( \tau_2 \) are close in value, the quadratic formula is better.