Problem 3.1b

Find the Laplace transform of $e^{-bt}\cos(\omega t)$ using the integral form of the equation.

Start with:

$$F(s) = \int_0^\infty e^{-bt} \cos(\varpi t) e^{-st}\, dt$$

We can combine the exponential terms to get:

$$F(s) = \int_0^\infty e^{-(b+s)t} \cos(\varpi t) dt$$

Replace the cos term with the exponential term with complex factors (equation 3-17):

$$F(s) = \int_0^\infty e^{-(b+s)t} \left(\frac{e^{j\omega t} + e^{-j\omega} t}{2} \right) dt$$

Now we'll multiply the terms in for the exponential and pull out the 1/2 to make it look neater:

$$F(s) = \frac{1}{2} \int_0^\infty \left(e^{-(b+s-j\omega)t} + e^{-(b+s+j\omega)t} \right) dt$$

Integrate both terms to get:

$$F(s) = \frac{1}{2}\left[-\frac{1}{b+s-j\omega}\left(e^{-(b+s-j\omega)\infty} - e^{-(b+s-j\omega)0}\right) - \frac{1}{b+s+j\omega}\left(e^{-(b+s+j\omega)\infty} - e^{-(b+s+j\omega)0}\right) \right]$$

The negative exponential terms go to zero and the exponential terms with 0 go to one to simplify to:

$$F(s) = \frac{1}{2}\left[-\frac{1}{b+s-j\omega}(-1) + -\frac{1}{b+s+j\omega}(-1) \right]$$

$$F(s) = \frac{1}{2}\left[\frac{1}{b+s-j\omega} + \frac{1}{b+s+j\omega} \right]$$

Now cross-multiply to get into one fraction:

$$F(s) = \frac{1}{2}\left[\frac{b+s+j\omega+b+s-j\omega}{b+s-j\omega(b+s+j\omega)} \right]$$

$$F(s) = \frac{1}{2}\left[\frac{2b+2s}{b^2+bs+bj\omega+sb+s^2+sj\omega-bj\omega-sj\omega-j^2\omega^2} \right]$$

Cancel the 1/2 with the 2 in the numerator. Then combine and simplify terms in the denominator:

$$F(s) = \left[\frac{b+s}{b^2+2bs+s^2+\omega^2} \right]$$

$$F(s) = \frac{b+s}{(b+s)^2+\omega^2}$$

Done!

Problem 3.10

Find the mathematical form, but not the constants for each of the following differential equations. Also use the final value theorem to determine the solution at long times.

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+8x=10$$

Take the Laplace transform of both sides. Recall that the initial condition and first derivative evaluated at time zero are zero.

$$s^2X(s)+4sX(s)+8X(s)=\frac{10}{s}$$

Combine all the X(s) terms and rearrange to get:

$$\left(s^2+4s+8\right)X(s)=\frac{10}{s}$$

$$X(s)=\frac{10}{s\left(s^2+4s+8\right)}$$

We need to do some factoring and partial fraction expanding to see what kind of partial fractions we're going to get. Let's complete the square on the polynomial portion: (see problem 3.6 if you are having a problem with completing the square)

$$s^2+4s+8=(s+2)^2+4$$

Now expand as partial fractions:

$$X(s)=\frac{10}{s\left(s^2+4s+8\right)}=\frac{A}{s}+\frac{Bs+C}{(s+2)^2+4}$$

We need to turn the second term into a format like equations 17 and 18 on page 47. We'll break this into two pieces with B(s+2) since we need the s+2 part, and then we'll have C-2B to add back in. Essentially we've added something in and subtracted it back out:

$$X(s)=\frac{10}{s\left(s^2+4s+8\right)}=\frac{A}{s}+\frac{B(s+2)}{(s+2)^2+4}+\frac{C-2B}{(s+2)^2+4}$$

$$X(s)=\frac{A}{s}+B\frac{s+2}{(s+2)^2+4}+\frac{C-2B}{\sqrt{4}}\frac{\sqrt{4}}{(s+2)^2+4}$$

Now invert using Table 3.1 to get:

$$x(t)=A+Be^{-2t}\cos(2t)+\frac{C-2B}{\sqrt{4}}e^{-2t}\sin(2t)$$

Now let's use the final value theorem on X(s) to find x as t goes to infinity:

$$\lim_{s\to 0} sX(s)=\frac{10s}{s\left(s^2+4s+8\right)}=\frac{10}{0+0+8}$$

We are going to repeat most of these steps when we do the rest of the problems so we won't describe as many of the steps in the solution unless there are unusual tricks. The general algorithm will be to:

1) take the differential equation into the Laplace domain
2) plug in any initial conditions
3) solve for X(s) on one side of an equation and everything else on the other side
4) do partial fraction expansion and try to get into forms we can invert
5) invert into the time domain.

b)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 10$$

$$s^2X(s) + 4sX(s) + 4X(s) = \frac{10}{s}$$

$$(s^2 + 4s + 4)X(s) = \frac{10}{s}$$

$$X(s) = \frac{10}{s(s^2 + 4s + 4)} = \frac{10}{s(s+2)(s+2)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$x(t) = A + Be^{-2t} + Cte^{-2t}$$

Using the final value theorem:

$$\lim_{s\to 0} sX(s) = \frac{10s}{s(s^2 + 4s + 4)} = \frac{10}{0+0+4} = \frac{10}{4}$$

c)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 10$$

$$s^2X(s) + 4sX(s) + 3X(s) = \frac{10}{s}$$

$$(s^2 + 4s + 3)X(s) = \frac{10}{s}$$

$$X(s) = \frac{10}{s(s^2 + 4s + 3)} = \frac{10}{s(s+3)(s+1)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$x(t) = A + Be^{-3t} + Ce^{-t}$$

$$\lim_{s\to 0} sX(s) = \frac{10s}{s(s^2 + 4s + 3)} = \frac{10}{0+0+3} = \frac{10}{3}$$

d)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 10$$

$$s^2X(s) + 4sX(s) + X(s) = \frac{10}{s}$$

$$\left(s^2 + 4s + 1\right)X(s) = \frac{10}{s}$$

$$X(s) = \frac{10}{s\left(s^2 + 4s + 1\right)}$$

On this one, we'll try to complete the square on the bottom to see what we get:

$$s^2 + 4s + 1 = (s+2)^2 - 3$$

Hmmm...It doesn't look good for us at this point because we have something that would require us to have $(s+b)^2 + \omega^2$. we have $+\omega^2 = -3$...Let's rethink this and try something else. How about just trying to factor our polynomial with the quadratic equation:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Whew! Well, at least we don't have any complex factors and we can write:

$$X(s) = \frac{10}{s\left(s^2 + 4s + 1\right)} = \frac{A}{s} + \frac{B}{s - 2 - \sqrt{3}} + \frac{C}{s - 2 + \sqrt{3}}$$

$$x(t) = A + Be^{-t\left(2+\sqrt{3}\right)} + Ce^{-t\left(2-\sqrt{3}\right)}$$

$$\lim_{s \to 0} sX(s) = \frac{10s}{s\left(s^2 + 4s + 1\right)} = \frac{10}{0+0+1} = \frac{10}{1} = 10$$

e)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = 10$$

$$s^2X(s) + 4sX(s) = \frac{10}{s}$$

$$\left(s^2 + 4s\right)X(s) = \frac{10}{s}$$

$$X(s) = \frac{10}{s\left(s^2 + 4s\right)} = \frac{10}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}$$

$$x(t) = A + Bt + Ce^{-4t}$$

$$\lim_{s \to 0} sX(s) = \frac{10s}{s\left(s^2 + 4s\right)} = \frac{10}{0+0+0} = \infty$$

f)

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}-5x=10$$

$$s^2X(s)+4sX(s)-5X(s)=\frac{10}{s}$$

$$\left(s^2+4s-5\right)X(s)=\frac{10}{s}$$

$$X(s)=\frac{10}{s\left(s^2+4s-5\right)}=\frac{10}{s(s+5)(s-1)}=\frac{A}{s}+\frac{B}{s+5}+\frac{C}{s-1}$$

$$x(t)=A+Be^{-5t}+Ce^{+t}$$

$$\lim\xrightarrow[s\to 0]{}sX(s)=\frac{10s}{s(s+5)(s-1)}=\frac{10}{0+0+-5}=-2?$$

No! Remember that we have to take the limit as s goes to zero from the positive side. If we try this, as s goes towards zero, the s - 1 term makes the equation undefined. This gives us a division by zero and the final value theory says that x(t) goes to infinity!.

g)

$$\frac{d^2x}{dt^2}-4\frac{dx}{dt}+3x=10$$

$$s^2X(s)-4sX(s)+3X(s)=\frac{10}{s}$$

$$\left(s^2-4s+3\right)X(s)=\frac{10}{s}$$

$$X(s)=\frac{10}{s\left(s^2-4s+3\right)}=\frac{10}{s(s-3)(s-1)}$$

$$X(s)=\frac{A}{s}+\frac{B}{s-3}+\frac{C}{s-1}$$

$$x(t)=A+Be^{3t}+Ce^{t}$$

If we apply the final value theorem here, we're going to end up with the same problem we had in the last part. Because we have (s - b) terms we will get an undefined term and the solution will blow up. We get x(t) goes to infinity.

h)

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 10$$

$$s^2X(s) - 4sX(s) + 4X(s) = \frac{10}{s}$$

$$\left(s^2 - 4s + 4\right)X(s) = \frac{10}{s}$$

$$X(s) = \frac{10}{s\left(s^2 - 4s + 4\right)} = \frac{10}{s(s-2)(s-2)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$x(t) = A + Be^{2t} + Cte^{2t}$$

Hmmm...I'm starting to see a pattern here. When we get s - b as a root (where b is a positive number), we get a solution that goes to infinity as time goes to infinity. This is going to come back much later on in this class. Negative roots lead to unstable solutions.

Problem 3.6

a) find x(t) for:

$$X(s)=\frac{s(s+1)}{(s+2)(s+3)(s+4)}=\frac{\alpha_1}{s+2}+\frac{\alpha_2}{s+3}+\frac{\alpha_3}{s+4}$$

We'll use Heaviside expansion. Multiply by s+2; let s = -2

$$X(s)=\frac{-2(-2+1)}{(-2+3)(-2+4)}=\alpha_1=\frac{-2(-1)}{(1)(2)}=1$$

Multiply by s+3, let s = -3:

$$X(s)=\frac{-3(-3+1)}{(-3+2)(-3+4)}=\alpha_2=\frac{6}{(-1)(1)}=-6$$

Multiply by s+4; let s = -4:

$$X(s)=\frac{-4(-4+1)}{(-4+2)(-4+3)}=\alpha_3=\frac{12}{(-2)(-1)}=6$$

Plug the constants is:

$$X(s)=\frac{1}{s+2}+\frac{-6}{s+3}+\frac{6}{s+4}$$

Then take the inverse Laplace transform of each term to get:

$$y(t)=e^{-2t}-6e^{-3t}+6e^{-4t}$$

b) find x(t)

$$X(s)=\frac{s+1}{(s+2)(s+3)(s^2+4)}=\frac{\alpha_1}{s+2}+\frac{\alpha_2}{s+3}+\frac{\alpha_3+\alpha_4 j}{s+2j}+\frac{\alpha_3-\alpha_4 j}{s-2j}$$

Notice that we had to use the complex conjugate because we are going to get imaginary roots from the s^2+4 term.
Now multiply by s+2, let s =-2:

$$X(s)=\frac{-2+1}{(-2+3)(-2^2+4)}=\alpha_1=\frac{-1}{(1)(8)}=-\frac{1}{8}$$

Multiply by s+3; let s = -3:

$$X(s)=\frac{-3+1}{(-3+2)(-3^2+4)}=\alpha_2=\frac{-2}{(-1)(13)}=-\frac{2}{13}$$

Now use Heaviside on the complex conjugate. Multiply by s+2j; let s = -2j:

$$\frac{-2j+1}{(-2j+2)(-2j+3)(-2j-2j)}=\alpha_3+\alpha_4 j$$

Recall that $s^2 + 4$ = (s+2j) (s-2j), which is what led to the -2j -2j term.

$$\frac{1-2j}{((-2j)^2-6j-4j+6)(-4j)}=\alpha_3+\alpha_4 j$$

$$\frac{1-2j}{(-4-10j+6)(-4j)}=\alpha_3+\alpha_4 j=\frac{1-2j}{(2-10j)(-4j)}=\frac{1-2j}{-40-8j}$$

We can't have any j's in the denominator so let's multiply the top and bottom by the complex conjugate (-40+8j):

$$\alpha_3 + \alpha_4 j = \frac{1-2j}{-40-8j}\frac{(-40+8j)}{(-40+8j)} = \frac{-40+8j+80j-16j^2}{1600-320j+320j-64j^2}$$

$$\alpha_3 + \alpha_4 j = \frac{-24+88j}{1600+64} = \frac{-24}{1664} + \frac{88}{1664}j$$

So, α_3 = -3/208 and α_4 = 11/208

$$X(s) = \frac{-\frac{1}{8}}{s+2} + \frac{\frac{2}{13}}{s+3} + \frac{\frac{-3}{208}+\frac{11}{208}j}{s+2j} + \frac{\frac{-3}{208}+\frac{11}{208}j}{s-2j}$$

Whew! Now use the table on pages 47-48 to find the inverse Laplace transforms of each term. For the complex roots, b = 0 and ω = 2:

$$x(t) = -\frac{1}{8}e^{-2t} + \frac{2}{13}e^{-3t} + 2\left(\frac{-3}{208}e^{0t}\cos(2t) + \frac{11}{208}e^{0t}\sin(2t)\right)$$

$$x(t) = -\frac{1}{8}e^{-2t} + \frac{2}{13}e^{-3t} + \frac{-3}{104}\cos(2t) + \frac{11}{104}\sin(2t)$$

c) find x(t) for:

$$X(s) = \frac{s+4}{(s+1)^2}$$

This has a repeating factor, which we discussed in class. Do partial fractions like:

$$X(s) = \frac{s+4}{(s+1)^2} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2}$$

We can do Heaviside by multiplying by $(s+1)^2$, let s = -1:

$$-1+4 = \alpha_2 = 3$$

But now we can't do Heaviside for α_1 because the left hand side and the α_2 term would be undefined. Let's choose a value for s and plug in α_2 = 3 to find α_1. Let's be clever and let s = -4 (the left hand side will equal zero then!)

$$X(s) = \frac{-4+4}{(s+1)^2} = 0 = \frac{\alpha_1}{-4+1} + \frac{3}{(-4+1)^2}$$

$$\frac{\alpha_1}{-3} = -\frac{3}{(-3)^2}$$

$$\alpha_1 = \frac{3}{3} = 1$$

So, we can plug the constants in to get:

$$X(s) = \frac{1}{s+1} + \frac{3}{(s+1)^2}$$

Now use table 3.1 on page 47-48 to find the inverse Laplace transform:

$$x(t) = e^{-t} + \frac{3te^{-t}}{(2-1)!} = (1+3t)e^{-t}$$

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\sqrt{\frac{3}{4}}t\right) = \frac{2\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

d) find x(t) for:

$$X(s) = \frac{1}{s^2 + s + 1}$$

There are a couple of different ways we could try to solve this one. We could use complex numbers much like we did before (and was done in last year's solution). However, this time, we'll complete the square and also find the constants that describe our system. First, let's complete the square. We need to find some way to make:

$$s^2 + s + 1 = (as + b)^2 + c$$

and find a, b, and c. Well, we will get s^2 by having a = 1. Now let's figure out what b should be. We know that when b = 1, we end up with a 2s term in the left hand side. So, why don't we try b = 1/2? Let's see what that gives us:

$$\left(s + \frac{1}{2}\right)^2 + c = s^2 + \frac{1}{2}s + \frac{1}{2}s + \frac{1}{4} + c$$

$$s^2 + s + \frac{1}{4} + c = s^2 + s + 1$$

We see from this expansion that c must be equal to 3/4. Not to bad, really. This brings up a general algorithm for completing the square:

1) find the a constant to give you the correct s2 term.
2) choose the value of b that gives you the correct factor in front of the s term
3) find the constant c to complete the square.

At this point, we have:

$$X(s) = \frac{1}{s^2 + s + 1} = \frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

This is looking like something we can find in Table 3.1. Equations 17 looks a lot like this, except we don't have ω^2 in the bottom and ω in the top. Here comes our trick...Let's multiply the top and bottom by the square root of our term:

$$X(s) = \frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} \times \frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{3}{4}}}$$

Now let's just rearrange this and pull out the terms we don't want in there while leaving it in the form of:

$$X(s) = \frac{\omega}{(s + b)^2 + \omega^2} \qquad\qquad X(s) = \frac{1}{\sqrt{\frac{3}{4}}} \frac{\sqrt{\frac{3}{4}}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Now take the inverse Laplace of this:

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\sqrt{\frac{3}{4}}t\right) = \frac{2\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

Problem 3.4 (a)

We first find a series of piecewise functions to describe f(t):

$$f(t)=\begin{cases}5 & 0<t<2\\ 1 & 2<t<6\\ 0 & t>6\end{cases}$$

Now we plug this into the definition of the Laplace transform:

$$F(s)=\int_0^2 5e^{-st}\,dt+\int_2^6 1e^{-st}\,dt+\int_6^\infty 0e^{-st}\,dt$$

We integrate each piece to get:

$$F(s)=5\left(\frac{-1}{s}\left(e^{-2s}-e^{-0s}\right)\right)+\left(\frac{-1}{s}\left(e^{-6s}-e^{-2s}\right)\right)+0$$

$$F(s)=\frac{-5e^{-2s}}{s}+\frac{5}{s}+\frac{-e^{-6s}}{s}+\frac{e^{-2s}}{s}$$

$$F(s)=\frac{1}{s}\left(5-4e^{-2s}-e^{-6s}\right)$$

Done with this one...

Problem 3.4 (b)

We are going to find the piecewise functions to describe f(t):

$$f(t)=\begin{cases}\frac{3}{2}t & 0<t<2\\ 3 & 2<t<6\\ 0 & t>6\end{cases}$$

Now plug this into the definition of the Laplace transform:

$$F(s)=\int_0^2 \frac{3}{2}te^{-st}\,dt+\int_2^6 3e^{-st}\,dt+\int_6^\infty 0e^{-st}\,dt$$

We can integrate the second and third terms like we did before. But now we need to use the chain rule for the first term: udv = uv - vdu.

$$u=t \qquad dv=e^{-st}$$
$$du=1 \qquad v=-\frac{1}{s}e^{-st}$$

$$F(s)=\frac{3}{2}\left(\left[-\frac{t}{s}e^{-st}\right]_0^2+\int_0^2\frac{1}{s}e^{-st}\,dt\right)+\frac{3}{s}\left(e^{-6s}-e^{-2s}\right)$$

$$F(s)=\frac{3}{2}\left(-\frac{2}{s}e^{-2s}-0+-\frac{1}{s^2}\left(e^{-2s}-e^{-0t}\right)\right)+\frac{3}{s}\left(e^{-6s}-e^{-2s}\right)$$

$$F(s)=-\frac{3}{s}e^{-2s}-\frac{3}{2s^2}e^{-2s}+\frac{3}{2s^2}+\frac{3}{s}e^{-6s}-\frac{3}{s}e^{-2s}=-\frac{3}{2s^2}e^{-2s}+\frac{3}{2s^2}-\frac{3}{s}e^{-6s}$$