Problem 4.12

Start with the equations from problem 3.20:

$$\frac{d(\rho V)}{dt} = w_i - w$$

$$\frac{d(\rho V C_a)}{dt} = w_i C_{ai} - w C_a - \rho V k C_a$$

In this problem, we are assuming that wi = w, so the first equation gives us that accumulation of total mass is zero. Now we will put the second equation into deviation variable form and expand the left hand side with the chain rule:

$$\rho V \frac{dC'_a}{dt} + C'_a \frac{d(\rho V)}{dt} = w_i C'_{ai} - wC'_a - \rho V k C'_a$$

a) But the second term on the left hand side is zero. Now take the Laplace of both sides and recall that w = w_i will be constant for this problem:

$$\rho V s C'_a(s) = w_i C'_{ai}(s) - wC'_a(s) - \rho V k C'_a(s)$$

Rearrange to get:

$$(\rho V s + w + \rho V k) C'_a(s) = wC'_{ai}(s)$$

We can now write our transfer function:

$$\frac{C'_a(s)}{C'_{ai}(s)} = \frac{w}{(\rho V s + w + \rho V k)}$$

b) We start with the same equation from above and take the Laplace. Everything that is a function of time becomes an (s) term:

$$\rho V s C'_a(s) = w(s) C'_{ai}(s) - w(s) C'_a(s) - \rho V k C'_a(s)$$

Now wait a minute...how are we going to get w(s) on one side and Ca(s) on the other? We can't! This is because our input and output are in the same term, making it so we cannot separate the equation into a transfer function. Recall that all differential equations must be linear (e.g., all coefficients must not be functions of time). Here, w is a function of time with the C_a term. This makes it nonlinear and we cannot get a transfer function.

Problem 5.3

$$\frac{Y'(s)}{X'(s)} = \frac{10}{s+1}$$

And, x(t) = 1 + t
y(0) = 2
x(0) = 1

We'll start by converting x(t) to deviation variables:

$$x' = x(t) - x_{ss}$$
$$x' = 1 + t - 1 = t$$

Now we'll take our x' into the Laplace domain:

$$X'(s) = \frac{1}{s^2}$$

Next, find Y'(s)and start the partial fraction expansion:

$$Y'(s) = \frac{10}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

Multiply by s^2, let s = 0 to get:

$$\frac{10}{(0+1)} = B = 10$$

Multiply by s+1 and let s = -1 to get:

$$\frac{10}{-1^2} = C = 10$$

We can't use the Heaviside method to get A so we'll let s = 1 and plug in B and C to get:

$$\frac{10}{1^2(1+1)} = \frac{A}{1} + 10 + \frac{10}{2}$$
$$10 - 10 - 5 = A = -5$$

So this give us:

$$Y'(s) = \frac{-5}{s} + \frac{10}{s^2} + \frac{10}{s+1}$$

We invert this back into the time domain to get:

$$y'(t) = -5 + 10t + 10e^{-t}$$

Now we'll convert out of deviation variable form and see that:

$$y(t) = y'(t) + y_{ss} = -5 + 10t + 10e^{-t} + 2$$
$$y(t) = -3 + 10t + 10e^{-t}$$

Problem 5.5

Mass = 1 gram
$C_p = 0.25$ cal/g °C
U = 20 cal/cm^2 h °C
A = 3 cm^2

a) We can do an energy balance on the thermocouple to get:

$$mC_p \frac{dT}{dt} = Q = UA(T_s - T)$$
$$\frac{dT}{dt} = \frac{UA}{mC_p}(T_s - T)$$
$$\frac{dT}{dt} = \frac{0.067}{\sec}(T_s - T)$$

Convert to deviation variables and then take the Laplace of both sides to get:

$$\frac{dT'}{dt} = \frac{0.067}{\sec}(T'_s - T')$$
$$sT'(s) = 0.067T'_s(s) - 0.067T'(s)$$

Rearrange to get:

$$\frac{T'(s)}{T'_s(s)} = \frac{0.067}{s + 0.067}$$

b) Now we put in a step change. Initially, our thermocouple is at 23°C. We abruptly plunge it into a bath at 80oC for 20 seconds. What is the temperature we read at the end of this time?

We'll put the deviation into deviation form first. $T'_s = T_s(t) - Ts_{(steady\ state)} = 80 - 23 = 57$ °C
Now convert this into the Laplace domain:

$$T'_s(s) = \frac{57}{s}$$

So,

$$T'(s) = \frac{57(0.067)}{s(s + 0.067)}$$

Now use partial fraction expansion to get:

$$T'(s) = \frac{57(0.067)}{s(s + 0.067)} = \frac{A}{s} + \frac{B}{s + 0.067}$$

Multiply by s, let s = 0:

$$\frac{57(0.067)}{(0 + 0.067)} = A = 57$$

Multiply by s + 0.067, let s = -0.067:

$$T'(s) = \frac{57(0.067)}{-0.067} = B = -57$$

This gives:

$$T'(s) = \frac{57}{s} + \frac{-57}{s + 0.067}$$

Invert this into the time domain:

$$T'(t) = 57 - 57e^{-0.067t}$$

Now take out of deviation variable format:

$$T(t) = T'(t) + T_{ss}$$

$$T(t) = 57 + 23 - 57e^{-0.067t} = 80 - 57e^{-0.067t}$$

We'll plug in out 20 seconds to get:

$$T(t) = 67.97^{o}C$$

Problem 5.6

A thermocouple was originally at 82°F and is suddenly put into a temperature of 105°F. After 1 second, the thermocouple reads 93°F. Now we put the same thermocouple into a stream where the temperature is falling at 2°F per second. What will the thermocouple be reading after 10 sec?

The gain of the thermocouple is one and it has first order dynamics so:

$$\frac{T'(s)}{T'_s(s)} = \frac{1}{\tau s + 1}$$

We will use the first piece of information to find out what the time constant is for this process:

$$T'_s(t) = 105 - 82 = 23$$

$$T'_s(s) = \frac{23}{s}$$

$$T'(s) = \frac{23}{s(\tau s + 1)}$$

Do the partial fraction expansion and find the constants:

$$T'(s) = \frac{23}{s(\tau s + 1)} = \frac{A}{s} + \frac{B}{\tau s + 1}$$

$$\frac{23}{(\tau(0) + 1)} = A = 23$$

$$T'(s) = \frac{23}{\frac{-1}{\tau}} = B = -23\tau$$

Now we'll invert the general solution back into the time domain:

$$T'(t) = 23 - \frac{23\tau}{\tau} e^{-\frac{t}{\tau}}$$

$$T'(t) = 23 - 23e^{-\frac{t}{\tau}}$$

$$T(t) = 105 - 23e^{-\frac{t}{\tau}}$$

We know that after 1 second, T(t) = 93°F

$$93 = 105 - 23e^{-\frac{1}{\tau}}$$

So, our time constant is 1.537

Now that we know the time constant, we can write:

$$\frac{T'(s)}{T'_s(s)} = \frac{1}{1.537s + 1}$$

Next, we put our thermocouple into an air stream that is dropping 2°F per second so the actual air temperature after 10 seconds will be 105 - 2(10) = 85°F

For this case, $T_s(t)$ = 105 - 2t and we convert this to deviation variable form:

$$T'_s(t) = 105 - 2t - 82 = 23 - 2t$$

Take our function into the Laplace domain and then get T'(s):

$$T'_s(s) = \frac{23}{s} - \frac{2}{s^2} = \frac{23s-2}{s^2}$$

$$T'(s) = \frac{23s-2}{s^2(1.537s+1)}$$

Do some rearranging to get:

$$T'(s) = \frac{23s-2}{s^2(1.537s+1)} = \frac{14.96s-1.301}{s^2(s+0.6506)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+0.6506}$$

This time, let's cross-multiply and combine like powers to get:

$$14.96s - 1.301 = As + 0.6506A + Bs^2 + 0.6506Bs + Cs^2$$

$$s^2: \quad 0 = B + C$$

$$s: \quad 14.96 = A + 0.6506B$$

$$s^0: \quad -1.301 = 0.6506A$$

Solving this system of equations leads to: A = -2, B = 26.068, C = -26.068
Plug these into our partial fraction expansion and then invert into the time domain:

$$T'(s) = \frac{-2}{s^2} + \frac{26.068}{s} + \frac{-26.068}{s+0.6506}$$

$$T'(t) = -2t + 26.068 - 26.068e^{-0.6506t}$$

$$T(t) = 108.068 - 2t - 26.068e^{-0.6506t}$$

So, after 10 seconds, the thermocouple will read 88.128^{o}F
The error in our reading is 88.128 - 85 = 3.128^{o}F

Problem 5.8

a) We need to use the figure given in the problem along with the following information:
$q_{1,ss}$ = 10 ft^3/min, $q_{2,ss}$ = 5 ft^3/min, h_{ss} = 4 ft.

The diameter of the tank is 6 ft in diameter, so the area is:

$$A = \pi\left(\frac{d}{2}\right)^2 = \pi\left(\frac{6\,ft}{2}\right)^2 = 28.27\,ft^2$$

The density of all streams is 60 lb/ft^3. Now let's set up our differential mass balance:

$$\frac{d(\rho V)}{dt} = q_1\rho + q_2\rho - q\rho$$

We can cancel out the densities because they are constant and equal. Then we can also replace V = hA where h is the height in the tank and A is the cross-sectional area:

$$\frac{d(Ah)}{dt} = \frac{Adh}{dt} = q_1 + q_2 - q$$

Now put everything into deviation variable format:

$$\frac{Adh'}{dt} = q'_1 + q'_2 - q'$$

Take the Laplace of the differential equation:

$$AsH'(s) = Q'_1(s) + Q'_2(s) - Q'(s)$$

When we are finding H'(s)/Q'$_1$(s), we can ignore the other terms (Q'$_2$(s) and Q'(s)):

$$\frac{H'(s)}{Q'_1(s)} = \frac{1}{As}$$

b) Now we need to deal with the input function. This looks a lot like what we did in Problem 5.4 so I'll skip most of the details and just begin with:

$$q'_1(t) = \begin{cases} 0 & t < 0 \\ 5 & 0 < t < 12 \\ 0 & t > 12 \end{cases}$$

Take the Laplace of this function:

$$Q'_1(s) = \int_0^{12} 5e^{-st}\,dt + \int_{12}^{\infty} 0e^{-st}\,dt$$

$$Q'_1(s) = \frac{5}{s}\left(1 - e^{-12s}\right)$$

Now we can plug this into our transfer function to find H'(s):

$$H'(s) = \frac{1}{28.27s}\left(\frac{5}{s}\left(1 - e^{-12s}\right)\right) = \frac{5}{s^2} - \frac{5}{s^2}e^{-12s}$$

We can use the same trick we did in 5.4 and separate the part we know from the one we don't. We can easily invert the first term as:

$$h'(t)(part\ one) = \frac{5}{28.27}t = 0.1769t$$

$$h'(t) = 0.1769t - [0.1769(t-12)]S(t-12)$$

Now time shift the second part like we did in lecture notes to get:

$$h(t) = \begin{cases} 4 + 0.1769t & 0 < t < 12 \\ 4 + 0.1769t - 0.1769(t-12) & t > 12 \end{cases}$$

Another way of rewriting this is as:

$$h'(t) = \begin{cases} 0.1769t & 0 < t < 12 \\ 0.1769t - 0.1769(t-12) & t > 12 \end{cases}$$

So, h(t) is h'(t) + h_{ss}:

c) Find the new steady state value of h(t)

We can either use the Final Value Theorem or we can evaluate this function at long times. We know that the new steady state occurs when t > 12 min. so we can use the second part of the time dependent solution. We then get:

$$h(t \to \infty) = 4 + 0.1769t - 0.1769t + 12(0.1769)$$
$$= 4 + 2.1228 = 6.1228\,ft$$