Problem 6.9

Given the following transfer function:

$$\frac{Y(s)}{X(s)} = \frac{1.5e^{-3s}}{(20s+1)(10s+1)}$$

For a step input of x(t) = 3, what is y(t)?

X(s) = 3/s. So,

$$Y(s) = \frac{4.5e^{-3s}}{s(20s+1)(10s+1)}$$

We'll use the Pade approximation to replace e^{-3s}:

$$e^{-3s} = \frac{1-\frac{3}{2}s}{1+\frac{3}{2}s} = \frac{2-3s}{2+3s}$$

$$Y(s) = \frac{4.5}{s(20s+1)(10s+1)}\frac{2-3s}{2+3s}$$

Now do the partial fraction expansion to get:

$$Y(s) = \frac{4.5}{s(20s+1)(10s+1)}\frac{2-3s}{2+3s} = \frac{A}{s}+\frac{B}{20s+1}+\frac{C}{10s+1}+\frac{D}{2+3s}$$

Then use Heaviside expansion to find the constants:

$$\frac{4.5}{(20(0)+1)(10(0)+1)}\frac{2-3(0)}{2+3(0)} = A = \frac{4.5(2)}{1(1)(2)} = 4.5$$

$$= \frac{4.5}{\frac{-1}{20}\left(10\frac{-1}{20}+1\right)}\left(\frac{2-3\frac{-1}{20}}{2+3\frac{-1}{20}}\right) = B = \frac{-90}{\frac{1}{2}}\left(\frac{\frac{43}{20}}{\frac{37}{20}}\right) = -180\frac{43}{37}$$

$$= \frac{4.5}{\frac{-1}{10}\left(20\frac{-1}{10}+1\right)}\frac{2-3\frac{-1}{10}}{2+3\frac{-1}{10}} = C = \frac{-45}{-1}\left(\frac{23}{17}\right) = 45\left(\frac{23}{17}\right)$$

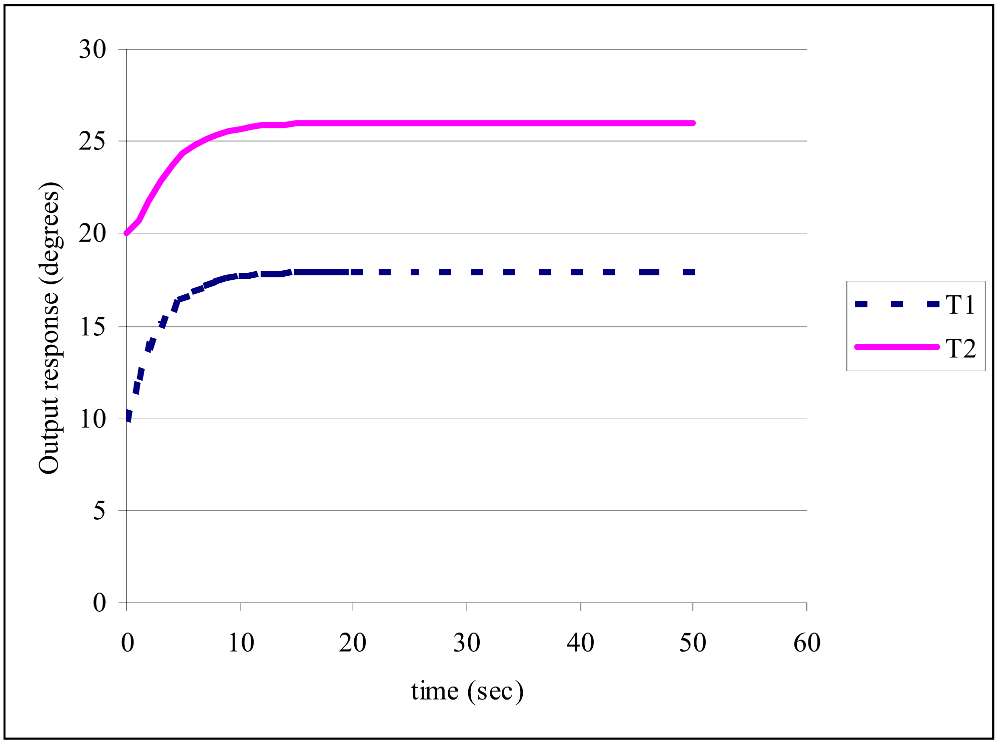
$$= \frac{4.5\left(2-3\frac{-2}{3}\right)}{\frac{-2}{3}\left(20\frac{-2}{3}+1\right)\left(10\frac{-2}{3}+1\right)} = D = \frac{6.75(4)}{\left(\frac{-40+3}{3}\right)\left(\frac{-20+3}{3}\right)} = \frac{243}{629}$$

So, Y(s) is

$$Y(s) = \frac{4.5}{s}+\frac{-180\frac{43}{37}}{20s+1}+\frac{45\frac{23}{17}}{10s+1}+\frac{\frac{243}{629}}{2+3s}$$

Problem 7.3

We start with the data given in the problem statement and make a plot of T1 and T2 versus time.



We have been asked to find a first order transfer function to describe the relationship between $T_1'(s)/Q'(s)$ and $T'_2(s)/T'_1(s)$.

We will set these parts up as:

$$\frac{T'_1(s)}{Q'(s)} = \frac{K_1}{\tau_1 s + 1}$$

$$\frac{T'_2(s)}{T'_1(s)} = \frac{K_2}{\tau_2 s + 1}$$

Recall from a recent homework and the notes that K = overall change in output/overall change in input in deviation variable form.
So,

$$K_1 = \frac{18-10}{84-82} = \frac{8}{2} = 4$$

$$K_2 = \frac{26-20}{18-10} = \frac{6}{8} = \frac{3}{4}$$

Recall that the second gain is the change in output from tank two divided by the input to tank two, which is really the output from tank 1. Now we need to find the time constants. We were told to assume that we did not have time delay, but the fractional response method on page 171-172 does have one. Well, let's go back to section 5.2 and use the same trick that we used before; i.e., $t = \tau$ when the response is 0.632 of the way to the new steady state value.

We need to find the time when:

$$0.632 = \frac{T_i(t) - 10}{18 - 10}$$
$$T_i(t) = 15.056$$
$$t = 3\,\text{min} = \tau$$

And,

$$0.632 = \frac{T_i(t) - 20}{26 - 20}$$
$$T_i(t) = 23.792$$
$$t = 4.1\,\text{min} = \tau$$

So, we get:

$$\frac{T'_1(s)}{Q'(s)} = \frac{4}{3s+1}$$
$$\frac{T'_2(s)}{T'_1(s)} = \frac{0.75}{4.1s+1}$$

b) Convert these transfer functions into differential equations now. We haven't done this before, but we can go backwards from our transfer function to our differential equation. Let's first separate our terms like this:

$$T'_1(s)(3s+1) = 4Q'(s)$$

Now rearrange and separate this out a bit:

$$3sT'_1(s) = 4Q'(s) - T'_1(s)$$

Invert into the time domain:

$$3\frac{dT'_1(t)}{dt} = 4q'(t) - T'_1(t)$$

Put everything into normal variables from deviation variables.

$$3\frac{dT_1(t)}{dt} = 4(q(t)+82) - (T_1(t)+10)$$

Here, we've chosen to explicitly show which things are functions of temperature.
Likewise,

$$4.1\frac{dT_2(t)}{dt} = 0.75(T_1(t)+10) - (T_2(t)+20)$$

These could have been rewritten a few ways, but we'll just stop here.

c) Instead of going back and trying to solve these differential equations with the differential equations methods, we'll use the transfer functions and the step change to solve for $T_1(t)$ and $T_2(t)$. First, $T_1(t)$,

$$q(t) = 84$$
$$q'(t) = 84 - 82 = 2$$
$$Q'(s) = \frac{2}{s}$$

So,

$$T'_1(s) = \frac{4(2)}{s(3s+1)} = \frac{\frac{4(2)}{3}}{s\left(s+\frac{1}{3}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{3}}$$

Do Heaviside expansion:

$$\frac{4(2)}{(3(0)+1)} = A = \frac{8}{1} = 8$$

$$\frac{\frac{4(2)}{3}}{-\frac{1}{3}} = B = -8\left(\frac{3}{3}\right) = -8$$

This gives us:

$$T'_1(s) = \frac{8}{s} + \frac{-8}{s+\frac{1}{3}}$$

Now invert into the time domain:

$$T'_1(t) = 8 - 8e^{-\frac{t}{3}}$$

Get out of deviation form by adding the steady state initial value:

$$T_1(t) = 18 - 8e^{-\frac{t}{3}}$$

Let's repeat these same steps for $T_2(t)$ now:

$$T'_2(s) = \frac{4(2)\frac{3}{4}}{s(3s+1)(4.1s+1)} = \frac{\frac{4(2)}{3}\frac{3}{4}\frac{1}{4.1}}{s\left(s+\frac{1}{3}\right)\left(s+\frac{1}{4.1}\right)} = \frac{\frac{2}{4.1}}{s\left(s+\frac{1}{3}\right)\left(s+\frac{1}{4.1}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{3}} + \frac{C}{\left(s+\frac{1}{4.1}\right)}$$

$$\frac{\frac{2}{4.1}}{\frac{1}{3}\left(\frac{1}{4.1}\right)} = A = 6$$

$$\frac{\frac{2}{4.1}}{\frac{-1}{3}\left(\frac{-1}{3}+\frac{1}{4.1}\right)} = B = 16.3636$$

$$\frac{\frac{2}{4.1}}{\frac{-1}{4.1}\left(\frac{-1}{4.1}+\frac{1}{3}\right)} = C = -22.3636$$

So,

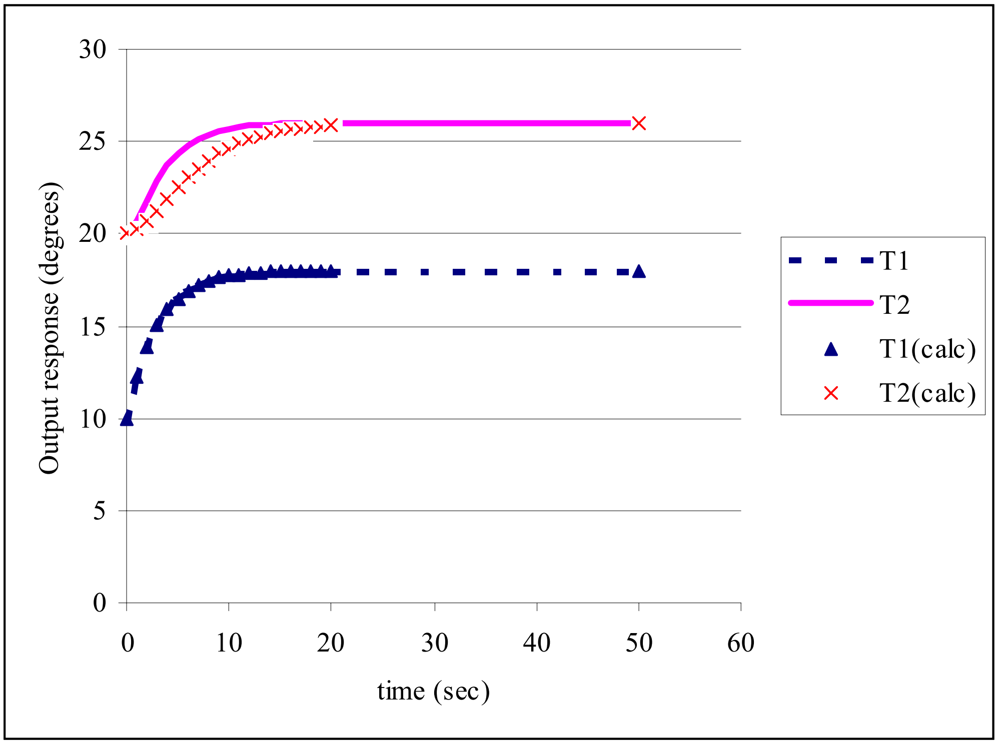
$$T'_2(s) = \frac{\frac{2}{4.1}}{s\left(s+\frac{1}{3}\right)\left(s+\frac{1}{4.1}\right)} = \frac{6}{s} + \frac{16.3636}{s+\frac{1}{3}} + \frac{-22.3636}{\left(s+\frac{1}{4.1}\right)}$$

Invert:

$$T'_2(t) = 6 + 16.3636e^{-\frac{t}{3}} - 22.3636e^{-\frac{t}{4.1}}$$

$$T_2(t) = 26 + 16.3636e^{-\frac{t}{3}} - 22.3636e^{-\frac{t}{4.1}}$$

Now that we have our two functions of time, we can plot them on the same plot we originally had.



You can see that our models do quite well at predicting the response curves.

Now invert this into the time domain:

$$y(t) = 4.5 - \frac{180}{20}\frac{43}{37}e^{-\frac{t}{20}} + \frac{45}{10}\frac{23}{17}e^{-\frac{t}{10}} + \frac{243}{629}\frac{1}{3}e^{-\frac{2}{3}t}$$

$$y(t) = 4.5 - 9\frac{43}{37}e^{-\frac{t}{20}} + \frac{207}{34}e^{-\frac{t}{10}} + \frac{81}{629}e^{-\frac{2}{3}t}$$

Let's do a quick check...This goes to 4.5 as time goes to infinity. The final value theorem shows:

$$\lim_{s\to 0}\left(\frac{4.5}{(20s+1)(10s+1)}\frac{2-3s}{2+3s}\right) = 4.5\frac{2}{2} = 4.5$$

We probably have the correct form of the solution since we have agreement between the final value and final value theorem.

Problem 7.4

We start with the following transfer function:

$$G(s) = \frac{2}{(6s+1)(4s+1)(2s+1)} = \frac{Y(s)}{X(s)}$$

Now a forcing function of a step input with a magnitude of 1.5 is added. So,

$$X(s) = \frac{1.5}{s}$$

We'll add this into our transfer function and rewrite the function a bit to get:

$$Y(s) = \frac{\frac{1}{4(2)}\frac{1}{2}}{s\left(s+\frac{1}{6}\right)\left(s+\frac{1}{4}\right)\left(s+\frac{1}{2}\right)} = \frac{\frac{1}{16}}{s\left(s+\frac{1}{6}\right)\left(s+\frac{1}{4}\right)\left(s+\frac{1}{2}\right)}$$

Do partial fraction expansion and find the constants that describe this system:

$$Y(s) = \frac{\frac{1}{16}}{s\left(s+\frac{1}{6}\right)\left(s+\frac{1}{4}\right)\left(s+\frac{1}{2}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{6}\right)} + \frac{C}{\left(s+\frac{1}{4}\right)} + \frac{D}{\left(s+\frac{1}{2}\right)}$$

$$\frac{\frac{1}{16}}{\left(0+\frac{1}{6}\right)\left(0+\frac{1}{4}\right)\left(0+\frac{1}{2}\right)} = A = \frac{(4)(2)6}{16} = \frac{6}{2} = 3$$

$$\frac{\frac{1}{16}}{\frac{-1}{6}\left(\frac{-1}{6}+\frac{1}{4}\right)\left(\frac{-1}{6}+\frac{1}{2}\right)} = B = \frac{\frac{-6}{16}}{\left(\frac{-2+3}{12}\right)\left(\frac{-1+3}{6}\right)} = \frac{(-3)(12)6}{(8)(1)(2)} = \frac{(-3)(3)3}{2} = \frac{-27}{2}$$

$$\frac{\frac{1}{16}}{\frac{-1}{4}\left(\frac{-1}{4}+\frac{1}{6}\right)\left(\frac{-1}{4}+\frac{1}{2}\right)} = C = \frac{\frac{-4}{16}}{\left(\frac{-3+2}{12}\right)\left(\frac{-1+2}{4}\right)} = \frac{(-1)(12)4}{(4)(-1)(1)} = \frac{(12)}{(1)} = 12$$

$$\frac{\frac{1}{16}}{\frac{-1}{2}\left(\frac{-1}{2}+\frac{1}{6}\right)\left(\frac{-1}{2}+\frac{1}{4}\right)} = D = \frac{\frac{-2}{16}}{\left(\frac{-6+2}{6(2)}\right)\left(\frac{-4+2}{4(2)}\right)} = \frac{(-1)(2)(6)(4)(2)}{(8)(-4)(-2)} = \frac{-(6)}{4} = -\frac{3}{2}$$

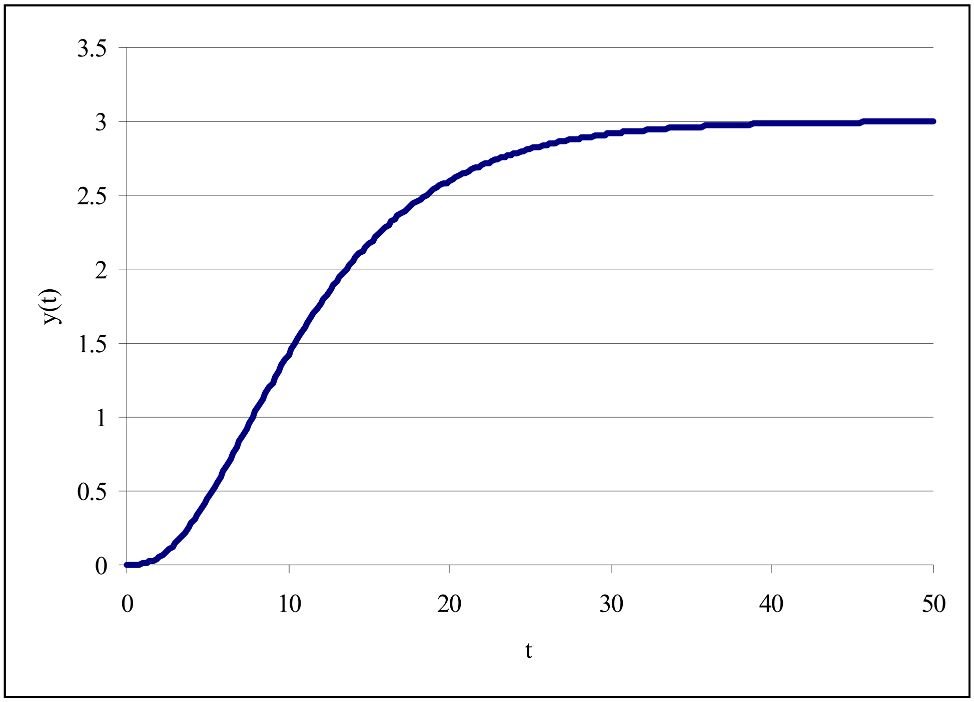
So, we plug these constants in to get:

$$Y(s) = \frac{3}{s} + \frac{\frac{-27}{2}}{\left(s+\frac{1}{6}\right)} + \frac{12}{\left(s+\frac{1}{4}\right)} + \frac{-\frac{3}{2}}{\left(s+\frac{1}{2}\right)}$$

Now we invert into the time domain to get:

$$y(t) = 3 - \frac{27}{2}e^{-\frac{t}{6}} + 12e^{-\frac{t}{4}} - \frac{3}{2}e^{-\frac{t}{2}}$$

Now let's plot this response versus t and see what we get:

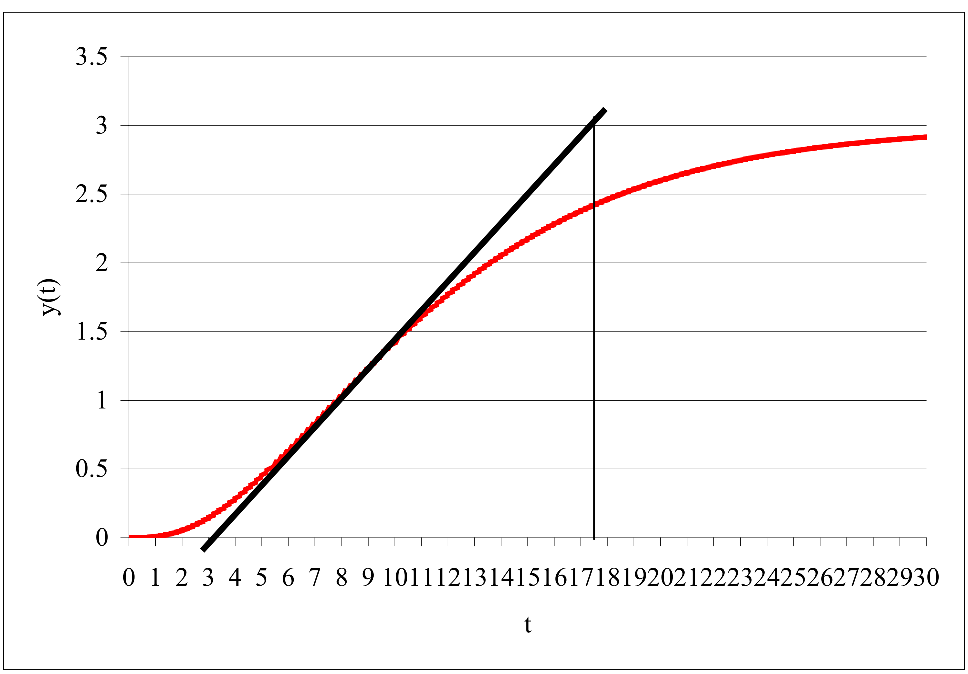


(a) We will now use the fraction incomplete response method to approximate this total complete solution as a first order + time delay transfer function:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

The steady state gain is found by:

$$K = \frac{3 - 0}{1.5 - 0} = 2$$



We can follow the algorithm on pages 171-172 to get that $\theta = 3$ and $\tau = 17.5\text{-}3 = 14.5$.

So,

$$G(s) = \frac{2e^{-3s}}{14.5s + 1}$$

b) We can also use the method of Harriot or Smith to find a second order transfer function like:

$$G(s) = \frac{K}{\left(\tau s^2 + 2\xi\tau s + 1\right)}$$

Again, K = 2.

We can use our data from our plot to find the time constants using Smith's method. The time when the response is 20% done is $t_{20} = 5.8$. And, $t_{60} = 12.2$ Using figure 7.7 we can find τ and ξ.

$$\frac{t_{20}}{t_{60}} = \frac{5.8}{12.2} = 0.475$$

So, $\xi = 0.7$ and $t_{60}/\tau = 1.65$
This leads to $\tau = 12.2/1.65 = 7.39$ So,

$$G(s) = \frac{2}{\left((7.39)^2 s^2 + 2(7.39)(0.7)s + 1\right)}$$