Problem 5.9

The diameter of the tank is 4 ft so, the area, A, is:

$$A = \Pi \left(\frac{d}{2} \right)^2 = 12.566 ft^2$$

q_1 = 8.33 h where q is in gal/min and h is in feet. This gives the 8.33 constant using of gal/ft min.
At steady state h = 6ft and q_i = 50 gal/min. At time zero, $q_i(t)$ = 70 gal/min. This gives:

$$q_i'(t) = 70 - 50 = 20 gal / \min$$

Let's solve system I first with an overall mass balance:

$$A\frac{dh}{dt} = q_i - q_1 = q_i - 8.33h$$

Divide by A and be careful of the units:

$$\frac{dh}{dt} = q_i \frac{1}{12.566 ft^2} \frac{ft^3}{7.84 gal} - 8.33 \frac{h}{A} \frac{gal}{ft \min} \frac{1}{12.566 ft^2} \frac{ft^3}{7.84 gal} ft$$

We now have everything in terms of ft/min now. We can rewrite this as:

$$\frac{dh}{dt} = 0.01015 q_i - 0.08455 h$$

And convert everything into deviation variables:

$$\frac{dh'}{dt} = 0.01015 q'_i - 0.08455 h'$$

Take the Laplace transform:

$$sH'(s) = 0.01015 Q'_i(s) - 0.08455 H'(s)$$

$$\frac{H'(s)}{Q'_i(s)} = \frac{0.01015}{s + 0.08455}$$

b) Now we'll find h(t) using the fact that $q'_i(t)$ = 20 gal/min. We'll take the Laplace transform of this input function first:

$$Q'_i(s) = \frac{20}{s}$$

We plug this into our transfer function and expand into partial fractions:

$$H'(s) = \frac{20(0.01015)}{s(s + 0.08455)} = \frac{A}{s} + \frac{B}{(s + 0.08455)}$$

Use Heaviside expansion. Multiply by s, let s = 0:

$$\frac{20(0.01015)}{(0 + 0.08455)} = A = 2.401$$

Multiply by s + 0.08455; let s = -0.08455:

$$\frac{20(0.01015)}{-0.08455} = B = -2.401$$

So, H'(s) is:

$$H'(s) = \frac{2.401}{s} + \frac{-2.401}{(s + 0.08455)}$$

Problem 5.18

a) we put in a ramp input at time zero with $q_i(t) = 1+0.4\ m^3/min * time$. Initially, $q_i = q_1 = q_2 = 1\ m^3/min$. Put the ramp input into deviation variable form:

$$q'_i(t) = 0.4t$$

Now let's find the overall transfer function:

$$\frac{Q'_1(s)}{Q'_i(s)} = \frac{1}{6s+1}$$

$$\frac{Q'_2(s)}{Q'_1(s)} = \frac{1}{6s+1}$$

So,

$$\frac{Q'_2(s)}{Q'_i(s)} = \left(\frac{1}{6s+1}\right)\frac{1}{6s+1} = \frac{1}{(6s+1)^2}$$

Let's take the Laplace of our forcing (input function):

$$Q'_i(s) = \frac{0.4}{s^2}$$

Plugging everything into the overall transfer function gives:

$$Q'_2(s) = \frac{1}{(6s+1)^2}\left(\frac{0.4}{s^2}\right)$$

Expand into partial fractions:

$$Q'_2(s) = \frac{1}{(6s+1)^2}\left(\frac{0.4}{s^2}\right) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{6s+1} + \frac{D}{(6s+1)^2}$$

Do Heaviside for B and D:

$$\frac{0.4}{(6s+1)^2} = B = \frac{0.4}{(6(0)+1)^2} = 0.4$$

$$\frac{0.4}{s^2} = D = \frac{0.4}{\left(\frac{-1}{6}\right)^2} = 36\left(\frac{4}{10}\right) = 36\left(\frac{2}{5}\right) = \frac{72}{5}$$

Now cross-multiply and plug in the two variables we already found:
Equate like powers:

$$0.4 - s = A\left((6s+1)^2 s\right) + \frac{4}{10}(6s+1)^2 + C(6s+1)s^2 + \frac{72}{5}s^2$$

$$0.4 - s = 36As^3 + 12As^2 + As + \frac{72}{5}s^2 + \frac{24}{5}s + \frac{4}{10} + 6Cs^3 + Cs^2 + \frac{72}{5}s^2$$

$$s^3: \quad 0 = 36A + 6C$$
$$s^2: \quad 0 = 12A + \frac{72}{5} + C + \frac{72}{5}$$
$$s^1: \quad -1 = A + \frac{24}{5}$$
$$s^0: \quad \frac{4}{10} = \frac{4}{10}$$

So,

$$A = \frac{-24}{5}$$
$$C = \frac{144}{5}$$

Plugging everything into our equation we get:

$$Q'_2(s) = \frac{\frac{-24}{5}}{s} + \frac{\frac{2}{5}}{s^2} + \frac{\frac{144}{5}}{6s+1} + \frac{\frac{72}{5}}{(6s+1)^2}$$

Invert into the time domain:

$$q'_2(t) = \frac{-24}{5} + \frac{2}{5}t + \frac{24}{5}e^{-\frac{1}{6}t} + \frac{2}{5}te^{-\frac{1}{6}t}$$

Now we take back out of deviation variable form:

$$q_2(t) = q'_2(t) + q_{2,s.s}$$
$$q_2(t) = \frac{-19}{5} + \frac{2}{5}t + \frac{24}{5}e^{-\frac{1}{6}t} + \frac{2}{5}te^{-\frac{1}{6}t}$$

Problem 5.14

$$Overshoot = \frac{12.7-11.2}{11.2} = 0.13393$$

And overshoot is also equal to:

$$Overshoot = \exp\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.13393$$

So we can solve for ξ:

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -2.0104$$
$$\zeta = 0.53902$$

The time for a period is 2.3 sec so we can find the time constant, τ:

$$2.3 = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$
$$\tau = 0.30833$$

K is the steady state gain and can be found to be:

$$K = \frac{11.2-8}{31-15} = 0.2\frac{mm}{psi}$$

We can combine everything to find that:

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.0951s^2 + 0.3324s + 1}$$

b) Rearrange the transfer function by cross-multiplying:

$$R'(s)0.0951s^2 + R'(s)0.3324s + R'(s) = 0.2P'(s)$$

Recall that:

$$Laplace\left[\frac{\partial^2 R'(t)}{\partial t^2}\right] = s^2 R'(s) - sf(0) - f'(0) = s^2 R'(s)$$
$$Laplace\left[\frac{\partial R'(t)}{\partial t}\right] = sR'(s) - f(0) = sR'(s)$$

So we can invert the top equation to get:

$$0.0951\frac{\partial^2 R'(t)}{\partial t^2} + 0.3324\frac{\partial R'(t)}{\partial t} + R'(t) = 0.2P'(t)$$
$$\frac{\partial^2 R'(t)}{\partial t^2} + 3.495\frac{\partial R'(t)}{\partial t} + 10.515R'(t) = 2.103P'(t)$$

And now take out of deviation variables where P = 15 psi, R = 8mm at time zero:

$$\frac{\partial^2 R(t)}{\partial t^2} + 3.495\frac{\partial R(t)}{\partial t} + 10.515R(t) - 84.12 = 2.103P(t) - 31.545$$

We can invert this into the time domain to get:

$$h'(t) = 2.401 - 2.401e^{-0.08455t}$$

c) Now we'll find h(t) as time goes to infinity. First convert out of deviation variable form:

$$h(t) = h'(t) + h_{ss}$$
$$h(t) = 8.401 - 2.401e^{-0.08455t}$$
$$t = -\infty$$
$$h(t) = 8.401\,ft$$

Note, we could have used the final value theorem here to get the same answer but this seemed easier since we were already asked to find the function.

d) Now we'll find the time when the height first goes over 8 ft:

$$h(t) = 8.401 - 2.401e^{-0.08455t} = 8$$
$$-2.401e^{-0.08455t} = -0.714$$
$$-0.08455t = -1.7897$$
$$t = 21.17\,\text{min}$$

We will now solve case II where q_1 is not a function of height. We are told that the system is initially at stead state which means:

$$\bar{q}_i = 50\,gal/\text{min}$$
$$\bar{q} = 50\,gal/\text{min}$$

This time our model becomes:

$$A\frac{dh}{dt} = q_i - 50$$

We'll divide by A and put in deviation variable form. Note that the second term, 50, is constant now and will drop out when we put it in deviation variable form:

$$\frac{dh'}{dt} = \frac{q'_i}{A} = 0.01015q'_i$$

We'll take the Laplace transform and plug in $Q'_i(s) = 20/s$ to get:

$$H'(s) = \frac{20(0.01015)}{s(s)} = \frac{0.203}{s^2}$$

b) We can invert this directly and then convert back out of deviation variable form to get:

$$h(t) = 6 + 0.203t$$

c) This solution goes to infinity as time goes to infinity.

d) Find t when h = 8 ft:

$$h(t) = 6 + 0.203t = 8$$
$$0.203t = 2$$
$$t = 9.85\,\text{min}$$

The end result is that the second tank overflows first.

Problem 6.1

Find the roots of the following transfer function:

$$G(s) = \frac{30}{24s^3 + 20s^2 + 10s + 2}$$

In this case, the steady state gain is 30.
To find the roots, set the denominator equal to zero and then use a root finding program or do a long division:

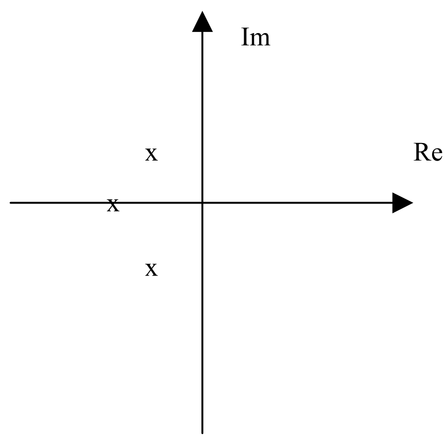
$$24s^3 + 20s^2 + 10s + 2 = 0$$

$$s^3 + \frac{20}{12}s^2 + \frac{5}{12}s + \frac{1}{12} = 0$$

$$\left(s + \frac{1}{3}\right)\left(s^2 + \frac{6}{12}s + \frac{1}{12}\right) = 0$$

$$\left(s + \frac{1}{3}\right)\left(s + \frac{1}{4} - \frac{\sqrt{3}}{4}j\right)\left(s + \frac{1}{4} + \frac{\sqrt{3}}{4}j\right) = 0$$

Matlab or Mathematica can be used to find these roots as well. We can now plot these roots in the Re-Im plane:



b) if x(t) is bounded for all time (i.e., it is not a ramping function), the general form of the solution will be:

$$y(t) = Ae^{-at} + Be^{-bt}\left(C\cos(ct) + D\sin(ct)\right)$$

The solution will be an oscillating exponential decay. All of the terms will decay and the process will be bounded for all time as long as x(t) is bounded.

Problem 6.9

Given the following transfer function:

$$\frac{Y(s)}{X(s)} = \frac{1.5e^{-3s}}{(20s+1)(10s+1)}$$

For a step input of x(t) = 3, what is y(t)?

X(s) = 3/s. So,

$$Y(s) = \frac{4.5e^{-3s}}{s(20s+1)(10s+1)}$$

We'll use the Pade approximation to replace e^{-3s}:

$$e^{-3s} = \frac{1-\frac{3}{2}s}{1+\frac{3}{2}s} = \frac{2-3s}{2+3s}$$

$$Y(s) = \frac{4.5}{s(20s+1)(10s+1)}\frac{2-3s}{2+3s}$$

Now do the partial fraction expansion to get:

$$Y(s) = \frac{4.5}{s(20s+1)(10s+1)}\frac{2-3s}{2+3s} = \frac{A}{s} + \frac{B}{20s+1} + \frac{C}{10s+1} + \frac{D}{2+3s}$$

Then use Heaviside expansion to find the constants:

$$\frac{4.5}{(20(0)+1)(10(0)+1)}\frac{2-3(0)}{2+3(0)} = A = \frac{4.5(2)}{1(1)(2)} = 4.5$$

$$= \frac{4.5}{\frac{-1}{20}\left(10\frac{-1}{20}+1\right)}\left(\frac{2-3\frac{-1}{20}}{2+3\frac{-1}{20}}\right) = B = \frac{-90}{\frac{1}{2}}\left(\frac{\frac{43}{20}}{\frac{37}{20}}\right) = -180\frac{43}{37}$$

$$= \frac{4.5}{\frac{-1}{10}\left(20\frac{-1}{10}+1\right)}\frac{2-3\frac{-1}{10}}{2+3\frac{-1}{10}} = C = \frac{-45}{-1}\left(\frac{23}{17}\right) = 45\left(\frac{23}{17}\right)$$

$$= \frac{4.5\left(2-3\frac{-2}{3}\right)}{\frac{-2}{3}\left(20\frac{-2}{3}+1\right)\left(10\frac{-2}{3}+1\right)} = D = \frac{6.75(4)}{\left(\frac{-40+3}{3}\right)\left(\frac{-20+3}{3}\right)} = \frac{243}{629}$$

So, Y(s) is

$$Y(s) = \frac{4.5}{s} + \frac{-180\frac{43}{37}}{20s+1} + \frac{45\frac{23}{17}}{10s+1} + \frac{\frac{243}{629}}{2+3s}$$

Now invert this into the time domain:

$$y(t) = 4.5 - \frac{180}{20}\frac{43}{37}e^{-\frac{t}{20}} + \frac{45}{10}\frac{23}{17}e^{-\frac{t}{10}} + \frac{243}{629}\frac{1}{3}e^{-\frac{2}{3}t}$$

$$y(t) = 4.5 - 9\frac{43}{37}e^{-\frac{t}{20}} + \frac{207}{34}e^{-\frac{t}{10}} + \frac{81}{629}e^{-\frac{2}{3}t}$$

Let's do a quick check...This goes to 4.5 as time goes to infinity. The final value theorem shows:

$$\lim_{s\to 0}\left(\frac{4.5}{(20s+1)(10s+1)}\frac{2-3s}{2+3s}\right) = 4.5\frac{2}{2} = 4.5$$

We probably have the correct form of the solution since we have agreement between the final value and final value theorem.