Problem 3.14

Solve the following differential equation:

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}+x=\sin t$$

where:

$$\frac{d^2x}{dt^2}=\frac{dx}{dt}=0$$

Take the Laplace transform to get:

$$s^2X(s)+sX(s)+X(s)=\frac{1}{s^2+1}$$

Rearrange to get X(s) by itself:

$$X(s)=\frac{1}{\left(s^2+1\right)\left(s^2+s+1\right)}$$

No do the partial fraction expansion:

$$X(s)=\frac{1}{\left(s^2+1\right)\left(s^2+s+1\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+s+1}$$

Cross-multiply to find like powers of s:

$$1=(As+B)\left(s^2+s+1\right)+(Cs+D)\left(s^2+1\right)$$

$$1=As^3+As^2+As+Bs^2+Bs+B+Cs^3+Cs+Ds^2+D$$

Now find the terms with like powers:

$$s^3: \quad 0=A+C$$

$$s^2: \quad 0=A+B$$

$$s^1: \quad 0=A+B+C$$

$$s^0: \quad 1=B+D$$

We can solve this system of equations to get: A = -1, B = 0. C = 1, and D = 1

$$X(s)=\frac{-s}{s^2+1}+\frac{s+1}{s^2+s+1}$$

We can invert the first term using Table 3.1. However, we need to manipulate the second term to be able to solve this one. Let's complete the square on the bottom first:

$$\frac{s+1}{s^2+s+1}=\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Now we'll separate the top part into two different terms:

$$\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

We see that we are almost there except that the second term still doesn't quite have the write numerator according to Table 3.1 Let's multiply by one (essentially) to get:

Problem 4.4

Start with the differential equation and find a transfer function relating Y(s) to X(s). In thi s problem, y and x are already deviation variables (this means that the functions and derivatives at time = zero are all zero.)

$$\frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+8\frac{dy}{dt}+4y=2\frac{dx}{dt}+3x$$

Take the Laplace of both sides and plug in the initial conditions:

$$s^3Y(s)+5s^2Y(s)+8sY(s)+4Y(s)=2sX(s)+3X(s)$$

Now separate and factor out the Y(s) and X(s) and rearrange to the final form:

$$\left(s^3+5s^2+8s+4\right)Y(s)=\left(2s+3\right)X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{\left(2s+3\right)}{\left(s^3+5s^2+8s+4\right)}$$

Problem 4.2

a) We'll start with the energy balance on the tank remembering that there is no flow in and no flow out:

$$mC_p \frac{dT}{dt} = Q_e$$

where Q_e is the heat being transferred from the heating element. A balance on the heating element then gives:

$$m_e C_{p_e} \frac{dT_e}{dt} = Q - Q_e = \frac{dQ_e}{dt}$$

We've written it in this form to make getting our final answer easier...

b) Now we take the Laplace of the equations to get:

$$mC_p sT'(s) = Q_e'(s)$$

$$Q'(s) - Q_e'(s) = sQ_e'(s)$$

Rearrange both equations to get:

$$\frac{Q_e'(s)}{Q'(s)} = \frac{1}{s+1}$$

$$\frac{T'(s)}{Q_e'(s)} = \frac{1}{mC_p s}$$

So,

$$\frac{T'(s)}{Q'(s)} = \frac{1}{mC_p s(s+1)}$$

c) Now let's check our units on the gain:]

$$K = \frac{1}{mC_p} [=] \frac{{}^oF\ lb}{Btu} \frac{1}{lb} = \frac{{}^oF}{Btu}$$

These are the correct units on the gain to relate a change in heat to a change in temperature.

d) We can look at equation 4-29 to see that:

$$\frac{T'(s)}{Q'(s)} = \frac{1}{wC_p} \frac{1}{b_2 s^2 + b_1 s + 1}$$

where:

$$b_2 = \frac{m m_e C_{p_e}}{w h_e A}$$

$$b_1 = \frac{m_e C_{p_e}}{h_e A} + \frac{m_e C_{p_e}}{wC_p} + \frac{m}{w}$$

The denominator of the second term with the b-terms could be rewritten as some combination of first order terms:

$$b_2 s^2 + b_2 s + 1 = (\tau_1 s + 1)(\tau_2 s + 1)$$

In our case with no flow, one of these first order terms has become an s term without the +1.

Problem 3.16

We already did most of the beginning of this problem in lecture but we'll sketch it out here again anyway. We have a tank with a flowrate q in and C_i in. The tank has a volume V_1 and an outlet flowrate q with a concentration C_1. This flows into a second tank with volume V_2, outlet flowrate q, and concentration C_2. This stream flows into another tank with volume V_3 and outlet flowrate q and concentration C_3.

a) We can write balances on the three tanks:

$$\frac{d(V_1C_1)}{dt} = V_1\frac{dC_1}{dt} = qC_i - qC_1$$

$$V_2\frac{dC_2}{dt} = qC_1 - qC_2$$

$$V_3\frac{dC_3}{dt} = qC_2 - qC_3$$

b) We now put in a step change of C_i into the first tank. We will find the form of $C_3(t)$ using Laplace transforms:

$$C_i(t) = h \quad for t > 0$$

$$C_i(s) = \frac{h}{s}$$

And,

$$V_1sC_1(s) = qC_i(s) - qC_1(s)$$

$$V_2sC_2(s) = qC_1(s) - qC_2(s)$$

$$V_3sC_3(s) = qC_2(s) - qC_3(s)$$

Rearrange and group terms:

$$C_1(s) = \frac{q}{V_1s+q}C_i(s)$$

$$C_2(s) = \frac{q}{V_2s+q}C_1(s)$$

$$C_3(s) = \frac{q}{V_3s+q}C_2(s)$$

Now plug in $C_i(s)$...Actually, we don't need to find C_1 or C_2 by themselves so we can just skip directly to $C_3(s)$:

$$C_1(s) = \frac{q}{V_1s+q}\frac{1}{s}$$

$$C_2(s) = \frac{q}{V_2s+q}\left(\frac{q}{V_1s+q}\right)\frac{1}{s}$$

$$C_3(s) = \frac{q}{V_3s+q}\left(\frac{q}{V_2s+q}\right)\left(\frac{q}{V_1s+q}\right)\frac{1}{s}$$

Now we need to invert into the time domain for V1 = V2 = V3=V:

$$C_3(s) = \frac{q}{Vs+q}\left(\frac{q}{Vs+q}\right)\left(\frac{q}{Vs+q}\right)\frac{1}{s} = \left(\frac{q}{Vs+q}\right)^3\left(\frac{1}{s}\right)$$

$$\left(\frac{q}{Vs+q}\right)^3\left(\frac{1}{s}\right) = \frac{A}{(Vs+q)^3} + \frac{B}{(Vs+q)^2} + \frac{C}{(Vs+q)^1} + \frac{D}{s}$$

This gives using Table 3.1 (some substitution and rearranging to get the V/q stuff:

$$C_3(t) = D + \frac{C}{V}e^{-\frac{q}{V}t} + \frac{\frac{B}{q}}{\left(\frac{V}{q}\right)^2 2}te^{-\frac{q}{V}t} + \frac{\frac{A}{q}}{\left(\frac{V}{q}\right)^3 6}t^2e^{-\frac{q}{V}t}$$

Now for the other part where none of the tank volumes are the same:

$$C_3(s) = \frac{q}{V_3s+q}\left(\frac{q}{V_2s+q}\right)\left(\frac{q}{V_1s+q}\right)\frac{1}{s} = \frac{A}{s} + \frac{B}{s+\frac{q}{V_3}} + \frac{C}{s+\frac{q}{V_2}} + \frac{D}{s+\frac{q}{V_1}}$$

Invert into the time domain:

$$C_3(t) = A + Be^{-\frac{q}{V_3}t} + Ce^{-\frac{q}{V_2}t} + De^{-\frac{q}{V_1}t}$$

Now for part c. If we don't know the total amount of tracer added to the first tank, can we still back calculated the amount from experimental data? Yes. Measure V_1, V_2, V_3, q, and all of the concentrations. The total amount of tracer will be equal to the amount in each tank plus the amount that has already left the system:

$$mass\,tracer = C_1V_1 + C_2V_2 + C_3V_3 + \int_o^t C_3 q\,dt$$

$$\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}}\frac{\sqrt{\frac{3}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Now we have:

$$X(s)=\frac{-s}{s^2+1}+\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}}\frac{\sqrt{\frac{3}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Invert into the time domain:

$$x(t)=-\cos(t)+e^{-\frac{1}{2}t}\cos\left(\sqrt{\frac{3}{4}}t\right)+\frac{\sqrt{3}}{3}e^{-\frac{1}{2}t}\sin\left(\sqrt{\frac{3}{4}}t\right)$$

As time goes to infinity, this solution will go to cos(t). The final value theorem should make this look like the solution goes to zero. Let's check:

$$\lim_{s\to 0} sX(s)=\frac{s}{\left(s^2+1\right)\left(s^2+s+1\right)}=0$$