Problem 11.8

We'll start this problem by writing out the transfer function for C(s)/R(s):

$$\frac{C(s)}{R(s)} = \frac{\dfrac{4.5(2)}{(10s+1)^2}}{1+\dfrac{4.5(2)}{(10s+1)^2}\left(\dfrac{1}{\tau_m s+1}\right)}$$

The bottom part of this transfer function is the characteristic equation, $1 + G_{ol}$. The characteristic equation is zero for stability.

$$1+G_{ol} = 0 = 1+\frac{4.5(2)}{(10s+1)^2}\left(\frac{1}{\tau_m s+1}\right)$$

We rearrange this and start simplifying to get:

$$0 = (10s+1)^2(\tau_m s+1)+9$$
$$(100s^2+20s+1)(\tau_m s+1)+9=0$$
$$100\tau_m s^3+20\tau_m s^2+\tau_m s+100s^2+20s+1+9=0$$
$$100\tau_m s^3+(20\tau_m+100)s^2+(\tau_m+20)s+10=0$$

For stability, all of the coefficients must be greater than zero. So,

$$\tau_m > 0$$

from the first term. The second and third terms do not add any thing more restrictive than this. Now let's do the Routh Stability test:

1	$100\tau_m$	τ_m+20
2	$20\tau_m+100$	10
3	$\dfrac{20\tau_m{}^2+500\tau_m+2000-1000\tau_m}{20\tau_m+100}$	0
4	0	

We see that we only get one more criteria here:

$$\frac{20\tau_m{}^2+500\tau_m+2000-1000\tau_m}{20\tau_m+100} > 0$$

Let's rearrange this and simplify it a bit to see what we get:

$$20\tau_m{}^2-500\tau_m+2000>0$$
$$\tau_m{}^2-25\tau_m+100>0$$

We'll use the quadratic formula to see what the roots of this are:

$$\tau_m = \frac{25\pm\sqrt{625-400}}{2}$$
$$\tau_m = 12.5\pm 7.5$$
$$\tau_m = 20 \text{ or } 5$$

I feel kind of silly that I didn't see that I could factor that our right away. So we have:

$$(\tau_m-20)(\tau_m-5)>0$$

This is the first time we've been confronted with two roots in the same inequality. For this inequality to be true, either both terms must be greater than zero or both terms must be less than zero (for the product to be positive). We already have the criteria that the time constant must be greater than zero so let's just look at the other criteria for both terms being positive:

$$\tau_m - 20 > 0$$
$$and \quad \tau_m - 5 > 0$$

Well, if τ_m is greater than 20, it is also greater than 5 so that would be our stability criteria.

$$\tau_m > 20$$

Another possibility is that both terms are less than zero. So, $0 < \tau_m < 5$ is also a criteria for stability. We would probably choose not to implement this one because we can make the process unstable by going to far in either direction.

b) Recall that the smaller the time constant is, the faster the response. If we were to apply our stability limit to a process with a time constant that is too small, we could add a transfer line or some other physical piece to our process to try to increase our time constant.

c) Now we are going to replace our proportional only controller with a PD controller. Remember from chapter 8 that adding derivative control has the potential of making our process unstable. Let's add that to our process to see what we get:

$$\frac{C(s)}{R(s)} = \frac{\dfrac{(4.5 + 4.5\tau_m s)(2)}{(10s+1)^2}}{1 + \dfrac{(4.5 + 4.5\tau_m s)(2)}{(10s+1)^2}\left(\dfrac{1}{\tau_m s + 1}\right)}$$

Our characteristic equation now becomes:

$$1 + G_{ol} = 0 = 1 + \frac{(4.5 + 4.5\tau_m s)(2)}{(10s+1)^2}\left(\frac{1}{\tau_m s + 1}\right)$$

Expand and combine terms to get:

$$100\tau_m s^3 + 20\tau_m s^2 + \tau_m s + 100s^2 + 20s + 1 + 9\tau_m s + 9 = 0$$
$$100\tau_m s^3 + (20\tau_m + 100)s^2 + (10\tau_m + 20)s + 10 = 0$$

Still, $\tau_m > 0$ is the criteria from the coefficients. The Routh test gives:

1	$100\tau_m$	$10\tau_m + 20$
2	$20\tau_m + 100$	10
3	$\dfrac{200\tau_m^{\,2} + 1400\tau_m + 2000 - 1000\tau_m}{20\tau_m + 100}$	0
4	0	

We can look at the bottom non-zero term and write:

$$\frac{200\tau_m^{\,2} + 400\tau_m + 2000}{20\tau_m + 100} > 0$$

So,

$$\tau_m^{\,2} + 2\tau_m + 10 > 0$$

Use the quadratic equation to get:

$$\tau_m = \frac{-2 \pm \sqrt{4-40}}{2}$$

$$\tau_m = \frac{-2 \pm 6j}{2}$$

$$\tau_m = -1 \pm 3j$$

We have imaginary roots! Can we still have a stable process? Yes, as long as we have $\tau_m > -1$ we will have an exponentially decaying process.

Guess what we forgot to do at this point...We forgot to check the last coefficient from our characteristic equation. So,

$$\tau_m > 0$$

We can have this be stable now.

Problem 11.14

We are starting with:

$$G_{ol} = \frac{K_c(1+\tau_D s)}{s^2(\tau s+1)}$$

a) We find the characteristic equation for proportional only control ($\tau_D = 0$):

$$1+G_{ol} = 1+\frac{K_c}{s^2(\tau s+1)} = 0$$

$$s^2(\tau s+1)+K_c = 0$$

$$\tau s^3 + s^2 + 0s + K_c = 0$$

We see that not all of the coefficients are greater than zero. Here, the s-term is zero so this can never be made to be stable.

b) Now we are asked to find the stability limits when τ_D is not zero:

$$1+G_{ol} = 1+\frac{K_c(1+\tau_D s)}{s^2(\tau s+1)} = 0$$

$$\tau s^3 + s^2 + K_c\tau_D s + K_c = 0$$

The coefficients tell us that:

$$\tau > 0$$

$$K_c\tau_D > 0$$

$$K_c > 0$$

Now we apply the Routh stability test to get:

1	τ	$K_c\tau_D$	0
2	1	K_c	0
3	$K_c(\tau_D-\tau)$	0	
4	K_c		

The only new criteria is that:

$$K_c(\tau_D-\tau) > 0$$

$$\tau_D > \tau$$

So our overall limits become:

$$\tau > 0$$

$$\tau_D > \tau$$

$$K_c > 0$$

Problem 11.9

We have a proportional only controller for a process with the following transfer function:

$$G(s) = \frac{e^{-2s}}{3s-1}$$

We will find the characteristic equation with the controller included:

$$1 + G_{ol} = 1 + \frac{K_c e^{-2s}}{3s-1} = 0$$

$$3s - 1 + K_c e^{-2s} = 0$$

Now we need to replace the exponential part of the equation with the Pade 1/1 approximation:

$$e^{-2s} = \frac{1-s}{1+s}$$

We plug this in and get:

$$3s - 1 + K_c \frac{1-s}{1+s} = 0$$

$$3s^2 + (2 - K_c)s + K_c - 1 = 0$$

This is only a quadratic equation so the Routh stability test won't give us any additional information beyond what we can get from the coefficients:

$$K_c - 1 > 0$$

$$2 - K_c > 0$$

So, $K_c > 1$ and $2 > K_c$
Our limits on stability are:

$$1 < K_c < 2$$

for a first order Pade Approximation.