ChEE 201
Fall 2005
University of Arizona
Computer Reading 6
Solving Linear Systems of Equations

At this point, we've covered how to do some computer programming, how to use infinite series, and then how to derive our own using Taylor series approximations. The Taylor series approximations can be used to transform non-linear equations into linear ones so we can solve them more easily. This section will discuss how to solve linear systems of equations using two methods: substitution and through combination of equations. This will set us up to be able to learn gaussian elimination in the next section. Once we learn gaussian elimination in that section, we'll go on to look at how a computer program could be used to solve systems of equations so we don't have to do quite as much work from then on.

<u>Learning Objectives:</u>
At the end of this section, students will be able to:
1) remember how to solve algebraic equations using substitution, including remembering to check their answer
2) understand why substitution is not the method of choice for solving equations when you have more than three variables and equations
3) be able to use the method of combining equations to solve systems of linear equations

<u>Substitution:</u>
Substitution is the way most people remember as a technique for solving linear systems of equations. This is when you solve one equation in terms of one of the variables and then plug that equation into the other equations to get rid of that variable. You keep successively doing this until you are left with one equation and one unknown. Then you move back up through the equations to find the rest of the variables. Let's look at an example using two equations with two unknowns:

$$2x + 3y = 4$$
$$x - 7y = 27$$

There are four different ways we could start this problem. We could solve the top equation for x, solve the top equation for y, solve the bottom equation for x, or solve the bottom equation for y. Because we are fairly clever engineers (and like to get away with the least amount of work possible), which one would you choose? I'd choose to solve the bottom equation for x to get:

$$x = 27 + 7y$$

Now we plug that equation back into our original top equation to get:

$$2x + 3y = 4$$
$$2(27 + 7y) + 3y = 4$$
$$54 + 14y + 3y = 4$$
$$50 = -17y$$
$$y = -\frac{50}{17}$$

Now that we have y, we can plug it into any of the equations to solve for x. We'll plug it into the equation that we had already rearranged in terms of x being by itself to get:

$$x = 27 + 7\left(\frac{-50}{17}\right) = \frac{27(17) - 350}{17} = \frac{459 - 350}{11} = \frac{109}{17}$$

When we finish, we should take both values and check them in at least one of the original equations to make sure that we haven't made any mathematical mistakes in our solution. Let's do that now using the original top equation:

$$2x + 3y = 4$$
$$2\left(\frac{109}{17}\right) + 3\left(\frac{-50}{17}\right) = 4$$
$$2(109) + 3(-50) = 68$$
$$218 - 150 = 68$$
$$68 = 68$$

Our answer checks out so we didn't make any mathematical mistakes. Students should always remember to check their answers in their equations when they are done to make sure they catch any problems.

Problems with Substitution:

Substitution is certainly a tried and true method for solving systems of linear equations. Let's look at a system of equations with three equations and three unknowns and try to apply substitution:

$$x + y + z = 6$$
$$6x + 3y - z = 9$$
$$-x - y + z = 0$$

Here, we have to solve one of the equations in terms of one of the variables and then plug that into the other two equations. Let's arbitrarily solve the top equation for x first:

$$x = 6 - y - z$$

Now we plug that into the other two equations to get two new equations. We'll do this side by side:

$$6(6 - y - z) + 3y - z = 9 \qquad -(6 - y - z) - y + z = 0$$
$$36 - 6y - 6z + 3y - z = 9 \qquad -6 + y + z - y + z = 0$$
$$-3y - 7z = -27 \qquad 2z = 6$$

Well, what do you know? We got a little lucky here in that we ended up with z by itself already. If we hadn't done the steps in the order we chose, we would now have to solve two equations with two unknowns and then be able to get to the answers. Let's wrap this one up, z = 3. Now plug that into the left equation above:

$$-3y - 7(3) = -3y - 21 = -27$$
$$-3y = -6$$
$$y = 2$$

Now, everything goes into our x equation to get:

$$x = 6 - 3 - 2 = 1$$

And what should we do now? We should go back to the top equations we started with and check at least one of them to make sure we did our mathematics correctly:

$$6x + 3y - z = 9$$
$$6(1) + 3(2) - 3 = 9$$
$$6 + 6 - 3 = 9$$
$$9 = 9$$

This one checks out correctly so we'll assume we did the problem correctly.

You notice that we had to do many more steps using substitution in order to reach an answer when we had three equations with three unknowns. Have you ever solved four equations with substitution? It takes even more calculations, and there is a much greater chance of making a mistake. Also, you use quite a bit of scrap paper up as you move through all the manipulations. The number of steps you have to do makes solving larger systems of equations with substitution just too tedious. Instead of relying on our tried and true method, we'll now talk about a way of combining equations to eliminate variables. You've probably seen this before but just may not have used it very much.

Combination of Equations:

We'll go back to our first system of equations presented in this section and recall what we started with:

$$2x + 3y = 4$$
$$x - 7y = 27$$

Let's look at the two equations and do a thought experiment. If we multiplied the second equation by -2, we could add it to the first equation to eliminate the x terms. This would look like:

$$\begin{array}{l} 2x+3y=4 \\ \underline{-2x+14y=-54} \\ 0x+17y=-50 \end{array}$$

So,

$$y=-\frac{50}{17}$$

We can then plug this into any of our other equations to find out what x is. We'll use the top equation and get:

$$\begin{aligned} 2x+3\left(\frac{-50}{17}\right)&=4 \\ 2x&=\frac{4(17)+3(50)}{17}=\frac{218}{17} \\ x&=\frac{109}{17} \end{aligned}$$

And, if we hadn't done our previous method to know what the answers are, we would now take our values we have and plug them into the original equations to make sure we haven't made any math mistakes.

If we have a larger system of equations, we can still choose what to do to carefully eliminate variables. We'll go back to our system of three equations and three unknowns and look at how that would come out, starting with:

$$\begin{array}{l} x+y+z=6 \\ 6x+3y-z=9 \\ -x-y+z=0 \end{array}$$

Let's add the first and third equations together:

$$\begin{array}{l} x+y+z=6 \\ \underline{-x-y+z=0} \\ 0x+0y+2z=6 \end{array}$$

This was even luckier than the first time we did the problem! We immediately got to one equation with one unknown. So, z = 3. Let's plug that into all three equations and rewrite them to get:

$$\begin{array}{lcl} x+y+3=6 & & x+y=3 \\ 6x+3y-3=9 & \Rightarrow & 6x+3y=12 \\ -x-y+3=0 & & -x-y=-3 \end{array}$$

Well, the first and third equations are now the same (note that this is true because we used those two to eliminate a variable the first time through). Now, we'll use the first and second equations, multiplying the first one by either minus 6 or minus 3 and adding it to the second equation. Let's do the multiplication by -3 to get:

$$\begin{array}{l} -3x-3y=-9 \\ \underline{6x+3y=12} \\ 3x+0y=3 \end{array}$$

And here we get to x = 1. We take this back up into one of the equations and we rapidly get y = 2. And, again, we'd check our answer in the original equations to make sure we didn't make any mistakes.

Let's do a more complex example and see how we would apply this method in a more rational manner. This first time we just seemed to randomly pick two equations and manipulate those as we worked through the equations. We'll do a 4 equation system next:

$$\begin{array}{l} 3x_1-2x_2-x_3+4x_4=12 \\ 2x_1+x_2-4x_3+x_4=-3 \\ -4x_1+2x_2+3x_3+2x_4=4 \\ x_1+3x_2+x_3-2x_4=-2 \end{array}$$

Our steps will be methodical in order to get rid of the x_1 terms first. We'll multiply the top equation by -2/3rds and add it to the second equation and then replace the second equation with that. This would look like:

$$-\frac{2}{3}(3x_1 - 2x_2 - x_3 + 4x_4 = 12)$$

$$\begin{array}{l} -2x_1 + \frac{4}{3}x_2 + \frac{2}{3}x_3 - \frac{8}{3}x_4 = -8 \\ \underline{2x_1 + x_2 - 4x_3 + x_4 = -3} \\ 0x_1 + \frac{7}{3}x_2 - \frac{10}{3}x_3 - \frac{5}{3}x_4 = -11 \end{array}$$

And plugging it into our system of equations we get:

$$\begin{array}{l} 3x_1 - 2x_2 - x_3 + 4x_4 = 12 \\ 0x_1 + \frac{7}{3}x_2 - \frac{10}{3}x_3 - \frac{5}{3}x_4 = -11 \\ -4x_1 + 2x_2 + 3x_3 + 2x_4 = 4 \\ x_1 + 3x_2 + x_3 - 2x_4 = -2 \end{array}$$

We can multiply the second row by 3 to get rid of the fractions for now so we'll do that:

$$\begin{array}{l} 3x_1 - 2x_2 - x_3 + 4x_4 = 12 \\ 0x_1 + 7x_2 - 10x_3 - 5x_4 = -33 \\ -4x_1 + 2x_2 + 3x_3 + 2x_4 = 4 \\ x_1 + 3x_2 + x_3 - 2x_4 = -2 \end{array}$$

Now, we'll multiply the top row by 4/3rds and then add it to the third row. Simultaneously, we'll multiply the top row by -1/3rd and add it to the bottom row to get:

$$\frac{4}{3}(3x_1 - 2x_2 - x_3 + 4x_4 = 12) \qquad -\frac{1}{3}(3x_1 - 2x_2 - x_3 + 4x_4 = 12)$$

$$\begin{array}{l} 4x_1 - \frac{8}{3}x_2 - \frac{4}{3}x_3 + \frac{16}{3}x_4 = \frac{48}{3} \\ \underline{-4x_1 + 2x_2 + 3x_3 + 2x_4 = 4} \\ 0x_1 - \frac{2}{3}x_2 + \frac{5}{3}x_3 + \frac{22}{3}x_4 = \frac{60}{3} \end{array} \qquad \begin{array}{l} -x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}x_4 = -4 \\ \underline{x_1 + 3x_2 + x_3 - 2x_4 = -2} \\ 0x_1 + \frac{11}{3}x_2 + \frac{4}{3}x_3 - \frac{10}{3}x_4 = -18 \end{array}$$

We'll multiply both of the resulting equations by 3 and plug them back into our list of equations

$$\begin{array}{l} 3x_1 - 2x_2 - x_3 + 4x_4 = 12 \\ 0x_1 + 7x_2 - 10x_3 - 5x_4 = -33 \\ 0x_1 - 2x_2 + 5x_3 + 22x_4 = 60 \\ 0x_1 + 11x_2 + 4x_3 - 10x_4 = -18 \end{array}$$

We'll continue on and multiply the second row by 2/7ths and add it to the third row while we will also multiply the second row by -11/7ths and add it to the fourth equation. Are you starting to see the pattern? To make it easier to follow, we'll now start leaving off the x_1 terms since they are zero.

$$\frac{2}{7}(7x_2 - 10x_3 - 5x_4 = -33) \qquad -\frac{11}{7}(7x_2 - 10x_3 - 5x_4 = -33)$$

$$2x_2 - \frac{20}{7}x_3 - \frac{10}{7}x_4 = -\frac{66}{7} \qquad -11x_2 + \frac{110}{7}x_3 + \frac{55}{7}x_4 = \frac{+363}{7}$$

$$-2x_2 + 5x_3 + 22x_4 = 60 \qquad +11x_2 + 4x_3 - 10x_4 = -6$$

$$0x_2 + \frac{15}{7}x_3 + \frac{144}{7}x_4 = \frac{354}{7} \qquad 0x_2 + \frac{138}{7}x_3 - \frac{15}{7}x_4 = \frac{237}{7}$$

$$0x_2 + 15x_3 + 144x_4 = +354 \qquad 0x_2 + 138x_3 - 15x_4 = 237$$

We plug this into our matrix of equations and we get:

$$\begin{aligned} 3x_1 - 2x_2 - x_3 + 4x_4 &= 12 \\ 0x_1 + 7x_2 - 10x_3 - 5x_4 &= -33 \\ 0x_1 + 0x_2 + 15x_3 + 144x_4 &= 354 \\ 0x_1 + 0x_2 + 138x_3 - 15x_4 &= 237 \end{aligned}$$

We're ready to work on the last two rows. We'll multiply the third row by -138/15 and add that equation to the bottom row:

$$\frac{-138}{15}(15x_3 + 144x_4 = 354)$$

$$-138x_3 - \frac{19872}{15}x_4 = -\frac{48852}{15}$$

$$138x_3 - 15x_4 = 237$$

$$0x_3 - \frac{20097}{15}x_4 = \frac{-45297}{15}$$

Our system of equations is now in the upper triangular form where all the coefficients are zero for the terms that lie to the lower left of the diagonal elements:

$$\begin{aligned} 3x_1 - 2x_2 - x_3 + 4x_4 &= 12 \\ 0x_1 + 7x_2 - 10x_3 - 5x_4 &= -33 \\ 0x_1 + 0x_2 + 15x_3 + 144x_4 &= 354 \\ 0x_1 + 0x_2 + 0x_3 - 20097x_4 &= -45297 \end{aligned}$$

So, now we can solve for x_4:

$$x_4 = \frac{45297}{20097} = 2.253918$$

We'll plug that result into the first three equations and rearrange to get:

$$\begin{aligned} 3x_1 - 2x_2 - x_3 &= 2.9843 \\ 0x_1 + 7x_2 - 10x_3 &= -21.73041 \\ 0x_1 + 0x_2 + 15x_3 &= 29.436 \end{aligned}$$

Again, we have one equation with one unknown for the last row now...$x_3 = 1.96238$. We plug this into the top two equations and simplify to get:

$$\begin{aligned} 3x_1 - 2x_2 &= 4.9467 \\ 0x_1 + 7x_2 &= -2.1066 \end{aligned}$$

This is almost like a zipper as we move up the equations. $x_2 = -0.30094$ And,

$$3x_1 = 4.3448$$
$$x_1 = 1.44827$$

Our solution, then, is: x_1 = 1.44827, x_2 =-0.30094, x_3 = 1.96238, x_4 = 2.25392. Now what should we do? You're absolutely correct if you said we should check our answers back in our orignal equations. We do and we find that:

We plug all of these numbers into the original top equation and we find that we do indeed have the solution.

You should be comfortable now using this method to get a system of linear equations into an upper triangular form. This is the basis of the gaussian elimination techniques we will be using in class to solve some of our homework problems. You are now ready to do Computer HW 6.